ON THE TRANSVERSE VIBRATIONS OF A REVOLVING TETHER

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Abstract. We analyse the transverse vibrations of a tether, modelled as an inextensible cable, and revolving at an average rate equal to the orbital rate. The reference motion is a revolving rigid tether. During this motion the force in the tether (time and location dependent) remains, in a first approximation, aligned with the tether axis. Separation of variables for the vibrations about this motion gives a Legendre equation for the spatial dependency of the deformations and Hill's equations for time dependency of the in- and out-of-plane deformations. The boundary conditions on the Legendre equation generate a series of admissible values of the separation constant that become equidistant. The two Hill's equations generate a series of intervals, contracting to equidistant critical values, where the solutions are unbounded. The admissible values of the separation constant must avoid these intervals. Asymptotic expressions for the separation constant and the critical values are given. The first and second in-plane deformation mode are unstable for zero end masses. By increasing the ratio of the concentrated over the distributed mass the deformation modes can be stabilised and the values of the separation constant can be made a multiple of the distribution of the critical points. Introducing unequal tip masses does not affect this result.

Key words: Revolving tether, cable dynamics, Legendre equation, Hill's equation, linear vibrations.

1. Notations and Abbreviations

1.1. Tether Data

l	tether length
$m_{1,2}$	concentrated tip masses of the tether
m	system mass ($= m_1 + m_2 + \sigma l$)
π_i	ratios of end masses and distributed mass to total mass
ρ	radius of gyration of the tether ($i = m\,\rho^2$)
σ	linear density of the tether
s	intrinsic length of the tether; origin at COM, takes the values $-s_1$ and s_2 at the end points, $s_1 = l(m_2 + \sigma l/2)/m$, $s_2 = l(m_1 + \sigma l/2)/m$
l_{ref}	reference length to obtain standard Legendre equation, $l_{\mathrm{ref}}^2 = s_1 s_2(2m_1 + 2m_2 + \sigma l)/\sigma l$
s_n	normalized intrinsic length s/l_{ref}
s_{ni}	values taken by s_i at the end points of the tether, $i = 1, 2$

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1.2. Elliptic functions

k modulus of elliptic functions
q nome of elliptic functions
$K(k)$ complete elliptic integral of the first kind
$E(k)$ complete elliptic integral of the second kind

1.3. Variables

n inertial rate of rotation of $\mathbf{R}_m(\dot{\vartheta}_i)$
n_c constant rate of rotation of the COM of the tether
n_K rate of rotation of a circular orbit at distance R, $n_K^2 = \mu/R^3$
$A_n(t)$ dimensionless function that multiplies the tension function $[f(s)]$ for the revolving tether. Reduces to 3 for a gravity gradient stabilised tether. $A_n(t) = \left[\sqrt{1 + 3/4\varepsilon^2} + \frac{\sqrt{3}}{k}\, dn(t)\right]^2 + 3\,\cos^2\,\alpha(t) - 1$
α in-plane angle from local vertical to tether axis
ε radius of gyration over distance to COM, ρ/R_{mc}
μ gravitational constant times mass of the attracting body
ν parameter in Legendre equation

1.4. Vectors

$\mathbf{f}$ dynamic inner force associated with the dynamic displacement (first order quantity)
$\mathbf{r}$ dynamic displacement from a reference motion (first order quantity), $\mathbf{r} = [x, y, z]^T$
$\mathbf{F}$ inner force in the tether (tension) at s
$\mathbf{F}_a$ applied force per length, in this application: $-\sigma\,\mu/R^2\,\mathbf{1}_r$
$\mathbf{R}$ radius vector to the material point s of the tether
$\mathbf{R}_{mc}$ radius vector of circular reference orbit
$\mathbf{1}_i$ unit vector of $\mathbf{R}_i$
κ_e curvature of the tether in the reference motion

Vectors are bold face, their modulus is plain text.

1.5. Abbreviations

COM Centre of mass
GGS gravity gradient stabilised
RHS right hand side
ODE ordinary differential equation
PDE partial differential equation

$\tilde{\mathbf{v}}$ tilde matrix corresponding to vector $\mathbf{v}$ (represents the cross product)

x	vectorial product or cross product
$(\ )'$	derivative w.r.t. the spatial variable s or s_n
$(\dot{\ })$	derivative w.r.t. time t or $\vartheta = n_K t$

2. Introduction

Breakwell (1987) and Gearhart (1990) studied the problem of expanding a circular orbit by having a tether revolving in the orbital plane with an average rate equal to the orbital rate. Modelling the tether as two equal point masses connected by a massless bar, the result is that such an orbit expansion is possible when J2 is included in the gravity potential. The tether length must be varied in an appropriate way. This note studies the stability of the revolving tether when it is modelled as an inextensible cable with zero bending stiffness.

The approach used for this vibrational problem of a one-dimensional continuum is described in Kulla (1982), Janssens and Crellin (1985). Spatial discretization of the tether is not needed. Loc Vu-Quo (1986) and Simo (1986, 1988) use a similar approach for applications to nonlinear problems. The motion of the rigid revolving tether as obtained by Breakwell and Gearhart (1987) is the reference motion for the vibrational problem. The tether motion is described by elliptic functions and the rotation rate varies between 0.44 and 1.788 times the orbital rate. Deviations of the COM of a circular reference orbit are neglected, Janssens (1990).

Separation of variables on the induced linear vibration problem leads to Legendre equations for the mode shapes and two Hill's equations for the time dependency of the in-plane and the out-of-plane deformations. The boundary conditions generate admissible values of the separation constant in the Legendre equation. These values must give bounded solutions for the two Hill's equations. This is investigated numerically and semi-analytically from asymptotic expressions for the separation constant and the critical values of the Hill's equations. The first two in-plane defor-mantion modes are unstable when the end masses are zero. They can be stabilised by adding large enough end masses.

3. Problem Formulation

As starting point we have one Kirchoff equation (balance equation) and an equation expressing the fact that in the inextensible cable model the inner force is tangent to the tether:

$$-\frac{d\mathbf{F}}{ds} + \sigma\,\frac{d\mathbf{R}^2}{dt^2} = \mathbf{F}_a \tag{1}$$

$$\frac{d\mathbf{R}}{ds} = \frac{\mathbf{F}}{|\mathbf{F}|}\,. \tag{2}$$

Equations 1 and 2 are a special case of a set of four equations describing a one-dimensional elastic continuum (Janssens, 1985; Simo, 1988). They will be linearized about the reference motion (R_{cm}, n_K) for the COM and $\alpha(t)$ about the COM:

$$n_K^2 = \mu / R_{cm}^3 \tag{3}$$

$$\dot{\alpha}(t) = \frac{\sqrt{3}\, n_K}{k}\, dn(\dot{\alpha}_0(0)t \mid k^2) \tag{4}$$

$$k^2 = \frac{3 n_K^2}{\dot{\alpha}_0(0)^2}\,, \qquad K(k)k = \sqrt{3}\, \frac{1}{\sqrt{1 + \frac{3}{4}\varepsilon^2}}\,. \tag{5}$$

For $\varepsilon = 0$: $k^2 = 0.938446\,,\ k = 0.968734$

$$F_e(s_n, t) = \frac{\sigma}{2}\, n_K^2\, l_{\text{ref}}^2\, A_n(t)(1 - s_n^2)\,. \tag{6}$$

By applying the procedure

$$\mathbf{R} = \mathbf{R}_e + \mathbf{r}\,, \qquad \mathbf{F} = \mathbf{F}_e + \mathbf{f} \qquad (\text{with } \mathbf{R}_e(s) = \mathbf{R}_{cm} + s\mathbf{1}_x)$$

and retaining only first order terms in $\mathbf{r}$, $\mathbf{f}$ independent of their magnitude compared to the equilibrium terms, which may be zero, the equilibrium terms cancel out and we obtain in a first step:

$$\frac{d\mathbf{f}}{ds} + \kappa_e \times \mathbf{f} = \sigma\, \frac{d\mathbf{r}^2}{dt^2} - \frac{d\mathbf{F}_a}{d\mathbf{R}}\,(\mathbf{R}_e)\mathbf{r} \tag{7}$$

$$\frac{d\mathbf{r}}{ds} + \kappa_e \times \mathbf{r} = \frac{\mathbf{f}}{F_e} - \frac{\mathbf{F}_e\mathbf{f}}{F_e^2}\, \frac{\mathbf{F}_e}{F_e}\,. \tag{8}$$

The terms $\kappa_e \times \mathbf{r}$ account for a possible dependency of the orientation of the equilibrium frame on s (equivalent to $\mathbf{V}_{\text{abs}} = \mathbf{V}_{\text{rel}} + \Omega\mathbf{x}\mathbf{r}$ with s as independent variable instead of t). The right-hand side of Equation 8 is rewritten as

$$-\frac{1}{F_e}\, \tilde{\mathbf{1}}_{fe}^2 \mathbf{f}$$

by using vector identities. Substituting these results in Equations 7–8 we have the following linear first order system of differential equations in s:

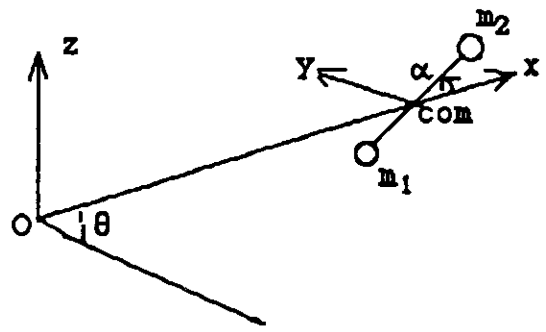


Fig. 1. Axis system.

$$\begin{bmatrix}\mathbf{r}\\ \mathbf{f}\end{bmatrix}' = \begin{bmatrix} -\kappa_e & -\frac{1}{F_e}\,\tilde{\mathbf{1}}^2_{fe} \\ \sigma\left[\frac{d(\,)^2}{dt^2} - \frac{d\mathbf{F}_a}{d\mathbf{R}}\right] & -\kappa_e \end{bmatrix} \begin{bmatrix}\mathbf{r}\\ \mathbf{f}\end{bmatrix} . \qquad (9)$$

The terms occurring in 9 will be represented in an axis coinciding with the rigid tether (Figure 1). The unit vector $\mathbf{1}_{fe}$ is along the x-axis, hence $\tilde{\mathbf{1}}^2_{fe} = \text{diag}\,\{0, -1, -1\}$, and κ_e is zero. The second order time derivative $d\mathbf{r}^2/dt^2$ in a frame rotating at $n_K + \dot{\alpha}$ is:

$$\frac{d\mathbf{r}^2}{dt^2} = \ddot{\mathbf{r}} + (\dot{n}_K + \ddot{\alpha})\mathbf{1}_z\mathbf{x}\mathbf{r} + 2(n_K + \dot{\alpha})\mathbf{1}_z\mathbf{x}\dot{\mathbf{r}} + (n_K + \dot{\alpha})^2\mathbf{1}_z\mathbf{x}(\mathbf{1}_z\mathbf{x}\mathbf{r}) \, . \qquad (10)$$

As the z-axis is perpendicular to the orbit plane the component equations corresponding to Equation 9 are easily obtained with $\mathbf{r}(s,t) = [x\,y\,z]^T$ the displacement of a material point of the tether from its location $[s\,0\,0]$ on a rigid tether. For the term $d\mathbf{F}_a/d\mathbf{R}$, the linearized gravity at R_{cm} is also expressed in this frame by a rotation α. Denoting the components of the dynamic force: $\mathbf{f}(s,t) = [f_x\,f_y\,f_z]^T$ we have for 9:

$$x' = 0$$

$$y' = \frac{2}{\sigma\, n_K^2\, l_{\text{ref}}^2\, A_n(t)(1 - s_n^2)}\, f_y$$

$$z' = \frac{2}{\sigma\, n_K^2\, l_{\text{ref}}^2\, A_n(t)(1 - s_n^2)}\, f_z$$

$$f'_x = \sigma\,[\ddot{x} - 2(n_K + \dot{\alpha})\dot{y} - (n_K + \dot{\alpha})^2 x - \ddot{\alpha} y] +$$
$$- \sigma\, n_K^2\,[(2\cos^2\alpha - \sin^2\alpha)x - 3\sin\alpha\cos\alpha\, y]$$
$$f'_y = \sigma\,[\ddot{y} - 2(n_K + \dot{\alpha})\dot{x} - (n_K + \dot{\alpha})^2 y - \ddot{\alpha} x] +$$
$$- \sigma\, n_K^2\,[(2\sin^2\alpha - \cos^2\alpha)y - 3\sin\alpha\cos\alpha\, x]$$
$$f'_z = \sigma\,[\ddot{z} + n_K^2 z]\,.$$

This set of equations decouples into an in-plane and an out-of-plane set. Combining terms we have:

$$x' = 0 \tag{11}$$
$$y' = \frac{2}{\sigma\, n_K^2\, l_{\text{ref}}^2\, A_n(t)(1 - s_n^2)}\, f_y \tag{12}$$
$$f'_x = \sigma\,[\ddot{x} - n_K^2\, A_n(t)x] - 2\sigma\,[\ddot{\alpha} y + (n_K + \dot{\alpha})\dot{y}] \tag{13}$$
$$f'_y = \sigma\,[\ddot{y} - n_K^2\,\{A_n(t) - 3\cos 2\alpha\}\, y] + 2\sigma(n_K + \dot{\alpha})\dot{x}) \tag{14}$$
$$z' = \frac{2}{\sigma\, n_K^2\, l_{\text{ref}}^2\, A_n(t)(1 - s_n^2)}\, f_z \tag{15}$$
$$f'_z = \sigma\,[\ddot{z} + n_K^2\, z]\,. \tag{16}$$

4. Separation of the Out-of-Plane Equations

The out-of-plane deflections $z(s,t)$ are described by the system 15, 16. Trying a solution of the type:

$$z(s,t) = Z(s)\, T_0(t) \tag{17}$$
$$f_z(s,t) = \Phi_z(s)\, n_K^2\, A_n(t)\, T_0(t) \tag{18}$$

we obtain

$$\Phi_z(s) = \frac{\sigma\, l_{\text{ref}}^2}{2}\,(1 - s_n^2)\, Z' \tag{19}$$
$$\{(1 - s_n^2)\, Z'\}' = -2c\, Z / l_{\text{ref}}^2 \tag{20}$$
$$\ddot{T}_0 + n_K^2\, T_0 = -c\, n_K^2\, A_n\, T_0 \tag{21}$$

where $-c$ is a separation constant. Changing the independent variable in 20 to s_n (derivations w.r.t. s_n are still denoted by $'$) it becomes a Legendre ODE in standard form with $2c = \nu(\nu + 1)$:

$$(1 - s_n^2)Z'' - 2s_n Z' + 2c\, Z = 0\,. \tag{22}$$

The differential equation for T_0 is a Hill's equation with $n_K^2[1 + c\, A_n(t)]$ as periodic function. Changing to the angular independent variable $\vartheta = n_K t$ it becomes (again with the same notation for derivatives w.r.t. ϑ):

$$\ddot{T} + P_0(\vartheta)T_0 = 0 \tag{23}$$

where

$$P_0(\vartheta\,;\, c) = 1 + c\, A_n\,. \tag{24}$$

The boundary conditions for 22 are $Z''(-s_{n1}) = Z''(s_{n2}) = 0$ when end masses are present. When the end masses go to zero the s_{ni} go to ± 1 and these conditions become $Z(\pm 1) =$ finite. These boundary conditions generate an infinite series of admissible values c_i. The solutions to 23 must be bounded for all these values. When both solutions are bounded none of them is, in general, periodic. Some facts on the Hill equation are collected in Appendix A. Hill's equation generates a series of intervals $[c_k - c'_k]$ for the separation constant where at least one of the solutions is unbounded. This series can be studied independently of the Legendre equation. These intervals go to zero with increasing c. The boundaries of these intervals are referred to as critical values. When the interval is so small that in practice the boundaries can be considered to be as coincident, the instability or divergence of the solutions is very slow. Everything happens as if there are two periodic solutions for that particular value of c. Hence the practical problem is to avoid these intervals until their width can be neglected. The determination of this limit depends on the application.

5. Separation of the In-Plane Equations

From 11 we see immediately that $x(t, s)$ can only depend on the time $x = x(t)$. Again assuming solutions:

$$y(s, t) = Y(s)\, T_i(t) \tag{25}$$

$$f_y(s, t) = \Phi_y(s)\, n_K^2\, A_n(t)\, T_i(t) \tag{26}$$

for Equations 12, 14 we have:

$$l_{\rm ref}^2\, \{(1 - s_n^2)Y'\}' = 2/\sigma\, \Phi_{y'} =$$

$$= 2\,[\ddot{T}_i - n_K^2\, \{A_n(t) - 3\, \cos\, 2\alpha\}\, T_i]\, \frac{Y}{n_K^2\, A_n\, T_i} + 4\, \frac{(n_c + \dot{\alpha})\dot{x}}{n_K^2\, A_n\, T_i}\,. \tag{27}$$

Neglecting for the moment the term in $\dot{x}$, a separation of variables is possible. With a separation constant $-c'$ and similar modifications as in the out-of-plane case:

$$(1 - s_n^2)Y'' - 2s_n Y' + 2c' Y = 0 \tag{28}$$

$$\ddot{T}_i + P_i(\vartheta)T_i = 0 \tag{29}$$

$$P_i(\vartheta\,;\,c') = -\Big(2\,\frac{\dot{\alpha}}{n_K} + \frac{3}{m}\Big) + c'\,A_n\,. \tag{30}$$

We again have a Legendre ODE with the same boundary conditions as in the out-of-plane case. This generates the same series of admissible values for the separation constant $c_i' = c_i$. This same series must give bounded solutions for the Hill's Equation 29. As the peridic function P_i differs from P_0 the sequence of critical intervals belonging to 27 will be different. However, the dependency of these functions on the separation constant c via A_n is the same. As shown in Appendix A, this implies that the critical values are asymptotically the same.

The in-plane equation contains a RHS proportional to $\dot{x}$. The assumption $\dot{x} = 0$ leads to an inhomogeneous Legendre equation. Assuming

$$\dot{x} = C\,\frac{AT_i}{4(n_c + \dot{a})}$$

gives a constant RHS ($= C$) for 27. This assumption may not lead to contradictions for f_x in 13. This point was not investigated further as it was assumed that x and f_x are only present in the rigid mode as for a rotating cable. The RHS does not play a role then in the calculation of the admissible values for the separation constant.

6. Legendre Equation – Mode Shapes

The Legendre Equations 22, 28 are in standard notation:

$$(1 - x^2)y'' - 2x\,y' + \nu(\nu + 1)y = 0\,. \tag{31}$$

The admissible values of the parameter ν in 31 are such that the the boundary conditions $y''(-s_{n1}) = y''(s_{n2}) = 0$ are satisfied. The mass properties of a given tether define the points $-s_{n1}$ and s_{n2} where the Legendre functions must be evaluated. Table I gives a summary of the cases covered.

For a massless tether the reference length becomes infinite. For both ends the Legendre functions should be evaluated at the same value zero. This is the limiting case where the tension is constant in the tether. The value of this constant is not given by Equation 6 as the product $l_{\text{ref}}\,s = 0 \cdot \infty$ is undefined. In the remaining cases the tether mass is taken into account. The tension in the tether is no longer constant. In first approximation it is maximum at the COM and decreases symmetrically around

TABLE I
Mass properties.

m_1	m_2	σ	m	π_1	π_2	π_d	$-s_1/\frac{l}{2}$	$s_2/\frac{l}{2}$	l_{ref}	$-s_{n1}$	s_{n2}
m_e	m_e	0	$2m_e$	$\frac{1}{2}$	$\frac{1}{2}$	0	1	1	inf.	0	0
m_1	m_2	0	m_1+m_2	$\frac{m_1}{m_1+m_2}$	$\frac{m_2}{m_1+m_2}$	0	π_2	π_1	inf.	0	
0	0	σ	σl	0	0	1	1	1	$\frac{l}{2}$	1	1
m_e	m_e	σ	$2m_e+\sigma l$	$\frac{m_e}{2m_e+\sigma l}$	$\frac{m_e}{2m_e+\sigma l}$	$\frac{\sigma l}{2m_e+\sigma l}$	1	1	$\frac{1}{2}\sqrt{1+\frac{4me}{\sigma}}$	$\frac{1}{l_{\text{ref}}}$	$\frac{1}{l_{\text{ref}}}$
0	m_2	σ	$m_2+\sigma l$	0	$\frac{m_2}{m_2+\sigma l}$	$\frac{\sigma l}{m_2+\sigma l}$	$2\pi_2+\pi_d$	π_d	$\sqrt{s_1 s_2}\,X$	$\frac{s_1}{l_{\text{ref}}}$	$\frac{2}{l_{\text{ref}}}$
m_1	0	σ	$m_1+\sigma l$	$\frac{m_1}{m_1+\sigma l}$	0	$\frac{\sigma l}{m_1+\sigma l}$	π_d	$2\pi_1+\pi_d$	"	"	"
m_1	m_2	σ	m	$\frac{m_1}{m}$	$\frac{m_2}{m}$	$\frac{\sigma l}{m}$	$2\pi_2+\pi_d$	$2\pi_1+\pi_d$	"	"	"

$X = \sqrt{2m_1 + 2m_2 + \sigma 1}$

it. For the revolving tether the tension in each point varies by the same factor at twice the orbital rate. *Including higher order terms in the gravity expansion or the exact gravity force makes this variation asymmetric along the tether length.* Such a model cannot result in a Legendre equation. When there are no end masses the tension becomes zero at the end points. When end masses are present the tension at the end points is in equilibrium with the force on the discrete masses.

The general solution y is a superposition of the two fundamental solutions: e_o, e_1. These solutions are calculated from the recurrence relation on the coefficients of their expansion at zero. The second derivatives are calculated in the same way. Let

$$y(x) = a\, e_0(x\,;\,\nu) + b\, e_1(x\,;\,\nu)\,. \tag{32}$$

Then we need the values of ν that make the following determinant zero:

$$\begin{vmatrix} e_0''(-s_{n1}\,;\,\nu) & e_1''(-s_{n1}\,;\,\nu) \\ e_0''(s_{n2}\,;\,\nu) & e_1''(s_{n2}\,;\,\nu) \end{vmatrix} = 0\,. \tag{33}$$

No simplifications were found by replacing the second derivatives using Equation 31 or replacing the derivatives by a combination of solutions using a different ν. When the end masses are equal $s_{n1} = s_{n2}$. As e_0 is an even function and e_1 an odd function, the same is true for their second derivative. Hence, $e_0''(x\,;\,\nu) = e_0''(-x\,;\,\nu)$, $e_1''(x\,;\,\nu) = -e_1''(x\,;\,\nu)$. Condition 33 reduces to $e_0''(s_{n2}\,;\,\nu) = 0$ and $e_1''(s_{n2}\,;\,\nu) = 0$ separately. The set of zeros of the determinant equation contains always 0 and 1. The remaining zeros are easily determined using their constant asymptotic separation (Janssens, 1985)

TABLE II
Admissible ν_i values for a tether with equal end masses.

$2\pi_1$	s_n	ν_2	ν_3	ν_4	ν_5	ν_6	ν_7	ν_8	$\Delta\nu$
0	1	2	3	4	5	6			1
.2	0.845	2.105	3.363	4.737	6.179	7.661			1.5601
.4	0.745	2.264	3.815	5.518	7.291	9.098			1.8676
.6	0.674	2.425	4.239	6.215	8.257	10.32			2.1230
.8	0.620	2.583	4.819	6.847	9.123	11.42			2.3481
1	0.5573	2.735	5	7.429	9.916	12.43	14.95	17.48	2.552
2	0.4472	3.416	6.564	9.859	13.20	16.56	19.93	23.30	3.388
4	0.3333	4.517	8.952	13.57	18.09	22.70	27.30	31.92	4.622
6	0.2773	5.418	10.85	16.38	21.94	27.52	33.09	38.67	5.589
8	0.2425	6.200	12.47	18.83	25.22	31.62	38.02	44.43	6.412
10	0.2182	6.899	13.91	21.01	28.13	35.26	42.39	49.52	7.140

$$\Delta\nu \sim \frac{\pi}{|\arccos(-s_{n1}) - \arccos(s_{n2})|} \,. \tag{34}$$

For equal end masses the asymptotic separation becomes:

$$\Delta\nu_e(s_n) \sim \frac{1}{\left|1 - \dfrac{\arccos(s_n)}{\pi/2}\right|} \,. \tag{35}$$

When the tip mass is zero the formula gives the exact result $\Delta\nu_e(0) = 1$ and $\nu_k = k$. This result was given by Breakwell and Andeen (1977a, b). When the equal end masses increase the spacing between the ν_k increases. The flexible frequencies move up and are at the limit pushed to infinity. The first flexible frequency (ν_2) becomes comparable to the asymptotic separation. In practice good estimates for ν_k are:

$$\boxed{\nu_k(s_n) \sim (k-1)\,\Delta\nu_e(s_n)} \tag{36}$$

for a wide range of the parameters. The smaller the variation of the tension is compared to the 'average' tension, the better this approximation is.

Equation 34 shows that $\Delta\nu$ attainable with unequal end masses are the same as the attainable with equal masses. In the sequel it will be interesting to have an expression for the end masses that realize the same $\Delta\nu$. By using the definitions of s_{n1}, s_{n2}, s_1, s_2 and l_{ref} we have for the ratio of each end mass to the distributed mass:

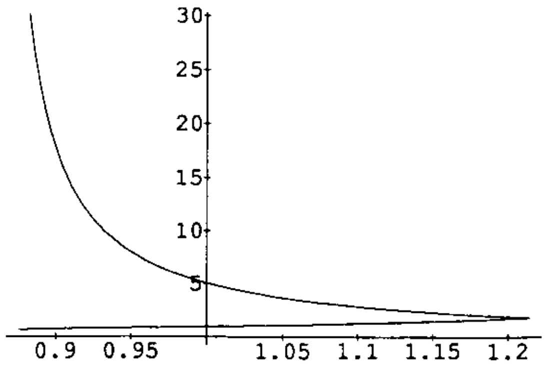


Fig. 2. Ratios of end mass over distributed mass needed to obtain an asymptotic separation of $\Delta\nu = 4.403$.

$$r_1 = \frac{m_1}{\sigma l} = \frac{1}{2(s_{n1} + s_{n2})}\left[\frac{1}{s_{n1}} - s_{n1}\right] \tag{37}$$

$$r_2 = \frac{m_2}{\sigma l} = \frac{1}{2(s_{n1} + s_{n2})}\left[\frac{1}{s_{n2}} - 2_{n2}\right] . \tag{38}$$

Fixing $\Delta\nu$ makes r_1 and r_2 functions of one parameter, e.g. $\vartheta_2 = \arccos s_{n2} < \pi/2$. As an example the unequal end masses r_1, r_2 needed to obtain $\Delta\nu = 4.40386$ are given in Figure 2. The corresponding ν_i are given in Table III. When mass $m_1 \rightarrow \infty$, $s_{n1} \rightarrow 0$ as the COM coincides with m_1. The mass m_2 goes to $\arccos(\pi/2 + \pi/\Delta\nu) = 0.654$ times the distributed mass to make $\Delta\nu$ the desired value.

7. Hill's Equation – Time Dependency

With ϑ as independent variable Equations 23 and 29 are of the type:

$$y'' + P(\vartheta; c)y = 0 \tag{39}$$

and P refers to any of the even periodic functions 24, 30 with period π. The transition matrix – corresponding to 39 in a first order formulation – after a period π is:

$$\Phi(\pi) = \begin{vmatrix} y_1(\pi) & y_2(\pi) \\ y_1(\pi)' & y_2(\pi)' \end{vmatrix} \tag{40}$$

where $\Phi(0) = E$ defines the initial conditions of the two independent fundamental solutions y_1, y_2.

TABLE III
Admissible ν_2 values for a tether with unequal end masses.

r_1	r_2	s_{n1}	s_{n2}	$\nu_{2,3,4,5}$
1.89112	1.71663	0.3359	0.362357	4.31756
				8.52588
				12.8586
				17.2261
2.28061	1.4633	0.28842	0.408487	4.31918
				8.5268
				12.8592
				17.2266
3.65305	1.09745	0.191388	0.49757	4.32885
				8.53244
				12.8631
				17.2295
5.05244	0.96023	0.142093	0.540302	4.33708
				8.54648
				12.8664
				17.232
10.2322	0.801728	0.072513	0.59783	4.35284
				8.54648
				12.8728
				17.2369
296.326	0.67195	0.002576	0.652438	4.37416
				8.55909
				12.8814
				17.2434

From Floquet theory it follows that the modulus of the eigenvalues λ_i of $\Phi(\pi)$ must be ≤ 1 for bounded solutions. The λ_i are the roots of:

$$\lambda^2 - \mathrm{tr}(\Phi)\lambda + \mathrm{Det}\,\Phi = 0\,. \tag{41}$$

Det Φ is obviously the Wronskian of Equation 39, hence Det $\Phi = 1$ and $\mathrm{tr}(\Phi) = y_1(p) + y_2(p)'$. *A priori* two integrations are needed to compute it. However, it is easily seen that y_1 is an even function and y_2 an odd function because P is even. Exploiting this feature when comparing $\Phi(\pi)^{-1} = \Phi(-\pi)$ we have:

$$\begin{vmatrix} y_2(\pi)' & -y_2(\pi) \\ -y_1(\pi)' & y_1(\pi) \end{vmatrix} = \begin{vmatrix} y_1(\pi) & -y_2(\pi) \\ -y_1(\pi)' & y_2(\pi)' \end{vmatrix}\,. \tag{42}$$

Hence, $y_2(\pi)' = y_1(\pi)$ and $\mathrm{tr}(\Phi) = 2y_1(\pi)$. The condition $\lambda_i \leq 1$ is now simply:

TABLE IV
Properties of $P_{i0}(\vartheta)$.

	Out-of-plane	In-plane
$P(\vartheta;\, c)$	$1 + 2c.$	$-(2\sqrt{3}/k\, dn + 3/m) + 2c.$
	$\{(0.5 + \sqrt{3}/k\, dn)^2 - 3/2m + 5/4\}$	$\{(0.5 + \sqrt{3}/k\, dn)^2 - 3/2m + 5/4\}$
$P_{max} = P(0)$	$1 + c\,\{3 + 3/m + 2\sqrt{3}/k\}$	$-(2\sqrt{3}/k + 3/m) + 2c.$
		$\{(0.5 + \sqrt{3}/k)^2 - 3/2m + 5/4\}$
	$1 + 9.7727\,c$	$-6.77268 + 9.7727\,c$
$P_{min} = P(K)$	$1 + c\,\{-3 + 3/m + 2\sqrt{3m_1}/k\}$	$-(2\sqrt{3m_1}/k + 3/m) + 2c.$
		$\{(0.5 + \sqrt{3m_1}/k)^2 - 3/2m + 5/4\}$
	$1 + 1.08395\,c$	$-4.08395 + 1.08395\,c$
$P_{average}$	$1 + 2c\,\{5 - 3/m + 6/m\, E(k)/K(k)\}$	$-(2 + 3/m) + 2c$
		$\{5 - 3/m + 6/m\, E(k)/K(k)\}$
	$1 + 1.08395\,c$	$-5.19677 + 1.08395\,c$

$m_1 = 1 - m$ and the argument of $dn(\,)$ is $\sqrt{3}/k\vartheta$.

$$\boxed{y_1(\pi) \le 1} \tag{43}$$

instead of the usual $\mathrm{tr}(\Phi) \le 2$. *One integration is sufficient to determine the stability of the solutions.* $\Phi(\pi)$ can be rewritten as:

$$\Phi(\pi) = \begin{vmatrix} y_1(\pi) & a \\ y_1(\pi)' & y_1(\pi) \end{vmatrix}, \quad a = \frac{\sqrt{y_1(\pi)^2 - 1}}{y_1(\pi)'} . \tag{44}$$

When $y_1(\pi) = 1$ there are periodic solutions with period π. The Floquet representation of the solutions gives no further information about the solution. We must distinguish the cases of Table V. The period of the functions is π. When $y_1(\pi) = -1$ the same table holds with a period 2π. The unstable solutions are related to the fact that a matrix of the type $\begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix}$ cannot be diagonalised. The full Jordan block form is needed. Table VI contains some evaluations of $y_1(\pi)$ for the out-of-plane periodic function. All the entries in this table are smaller than 1. The bold values correspond to the $c-$ values for a tether without end masses. The corresponding fundamental solutions are plotted in Figures 3–14. When $y_1(\pi) < 1$ it is the real part of the eigenvalues of $\Phi(\pi)$. *When the angle φ defined by $\phi = \arccos y_1(\pi)$ is an exact divisor of π, $\varphi = \pi/d$ then y_1 is a periodic solution with period $d\,2\pi$.* This is the case for $c = 1$: $\varphi_1 = \pi/6$ up to the numerical precision when doing more

TABLE V
Type of solutions for $y_1(\pi) = 1$.

$y_1(\pi)$	$y_1(\pi)'$	a	$\Phi(\pi)$	y_1	y_2
1	$\neq 0$	0	$\begin{vmatrix} 1 & 0 \\ y_1' & 1 \end{vmatrix}$	unstable	periodic odd
1	0	$\neq 0$	$\begin{vmatrix} 1 & a \\ 0 & 1 \end{vmatrix}$	periodic even	unstable
1	0	0	$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$	periodic even	periodic odd

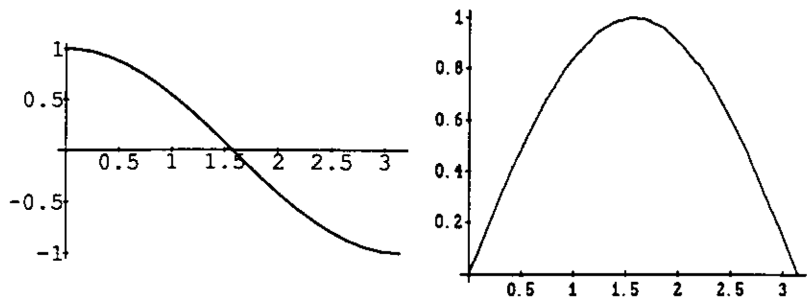
TABLE VI
$y_1(\pi)$ values out-of-plane.

c	+0.00	+0.25	+0.50	+0.75
1	**0.866027**	9.38575	−0.196978	−0.683096
2	−0.957756	−0.987358	−0.799937	−0.46083
3	**−0.053866**	0.353866	0.688568	0.910853
4	0.999957	0.955532	0.793837	0.542853
5	0.237152	−0.0868625	−0.394949	−0.658054
6	**−0.854287**	−0.969916	−0.999472	−0.945163
7	−0.815751	−0.625096	−0.390529	−0.131206
8	0.133439	0.385055	0.607484	0.787588
9	0.915785	0.986305	0.997188	0.950055
10	**0.849717**			

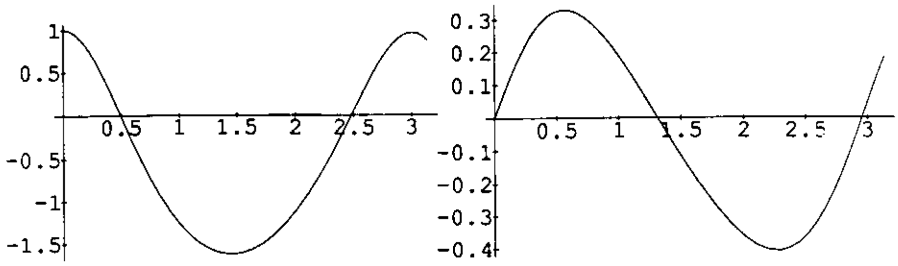
precise integrations (Loc Vu-Quoc, 1986). The first flexible mode has a period of six orbital revolutions. The total phase angle of the solution over $\pi = 1/2$ orbit period is $13/12$ and $13.(2\pi)$ over six orbit periods. The remaining figures allow to determine φ in the interval $0 - 2\pi$ instead of in the interval $0 - \pi$. For the next mode $c = 2$ the phase angle is *approximately* $7/4\pi$ per half orbit period. If this relation were exact there would be 21 periods in six orbits. The same pattern of an increase by eight periods over six orbit periods holds with decreasing accuracy for the higher modes.

The values $y_1(\pi)$ as a function of c are represented in Figure 18. They contain intervals of Table VII where $y_1(\pi) > 1$.

At each boundary c_{low} or c_{high} there is a periodic solution (one even and one odd) while the other is unbounded. If the interval goes to zero the periodic solutions are coexistant and there is no stability problem. At these 'critical points' there are

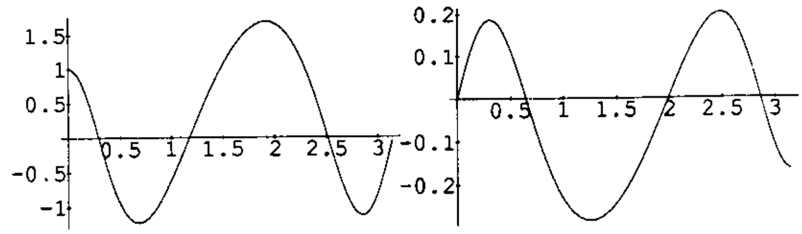


Figs. 3, 4. Out-of-plane fundamental solutions for $\nu = 0$, $c = 0$.

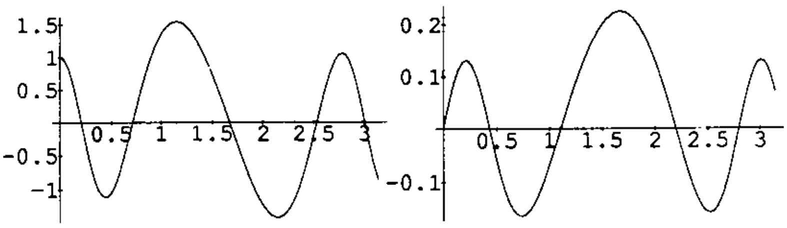


Figs. 5, 6. Out-of-plane fundamental solutions for $\nu = 1$, $c = 1$.

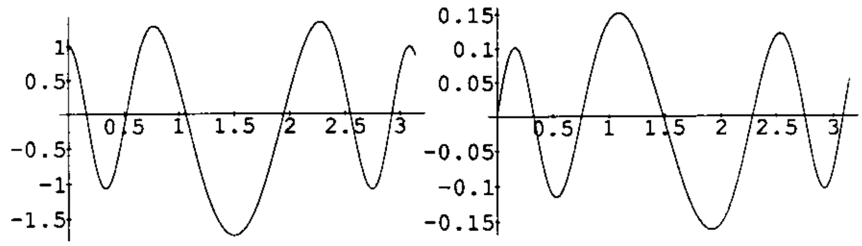
two periodic solutions with the same period (π or 2π). Numerically the instability intervals seem to disappear. If this is truly the case or not is a difficult question.



Figs. 7, 8. Out-of-plane fundamental solutions for $\nu = 2$, $c = 3$.

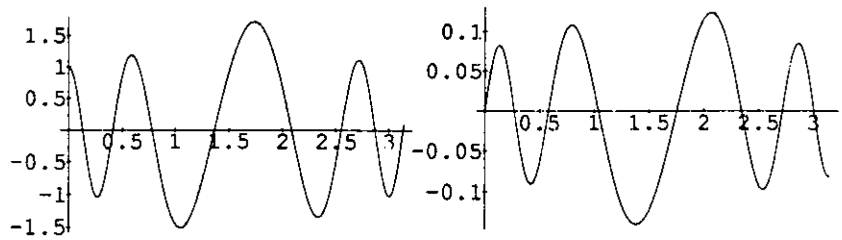


Figs. 9, 10. Out-of-plane fundamental solutions for $\nu = 3$, $c = 6$.

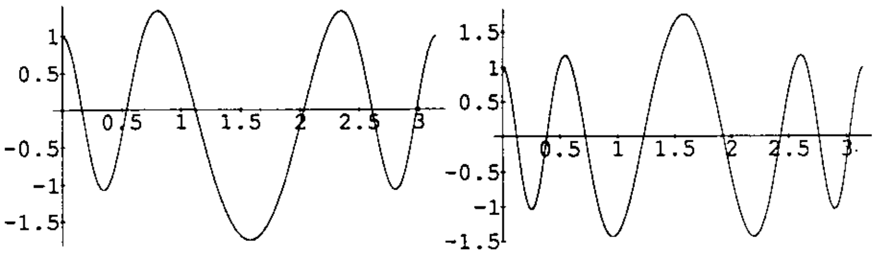


Figs. 11, 12. Out-of-plane fundamental solutions for $\nu = 4$, $c = 10$.

question. The answer is known for some $P(\vartheta)$ functions in Hill's equation. For practical applications the time constant to double the amplitude increases quickly when $y_1(\pi)$ barely exceeds one. Taking 1.5 h as a lower bound for the orbit period



Figs. 13, 14. Out-of-plane fundamental solutions for $\nu = 5$, $c = 15$.



Figs. 15, 16. Even periodic solution – period π – for $c = 9.43, 16.9$ (P_0).

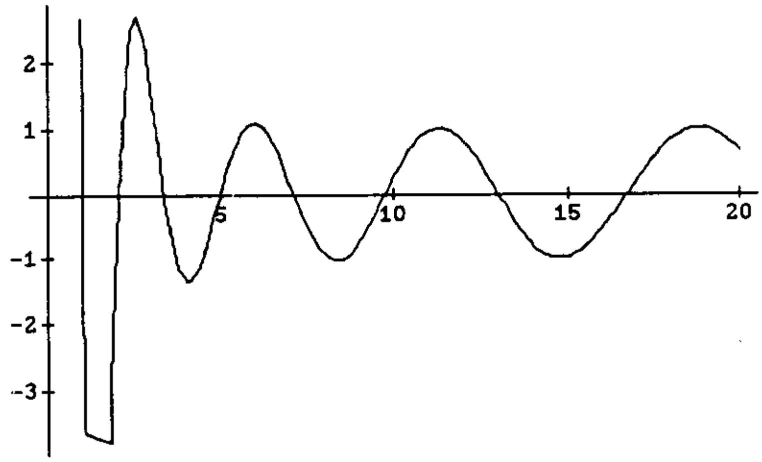


Fig. 17. $y_1(\pi)$ versus separation constant c, in-plane (P_i).

we have:

$$y_1(\pi)_{\max} = 1.001 \Rightarrow \text{Time constant} \sim 22 \text{ days}$$
$$y_1(\pi)_{\max} = 1.0001 \Rightarrow \text{Time constant} \sim 217 \text{ days}$$
$$y_1(\pi)_{\max} = 1.00001 \Rightarrow \text{Time constant} \sim 6 \text{ years} .$$

Depending on the duration of the mission one can require that $y_1(\pi)_{\max}$ is below a certain value for the admissible ν_i. This requirement gives a bound for ν below which the instability intervals must be avoided. Notice also that difference in ν_i values has stabilised on $\Delta\nu = 0.735$ and $\nu_k \sim k\,\Delta\nu + 0.1987$.

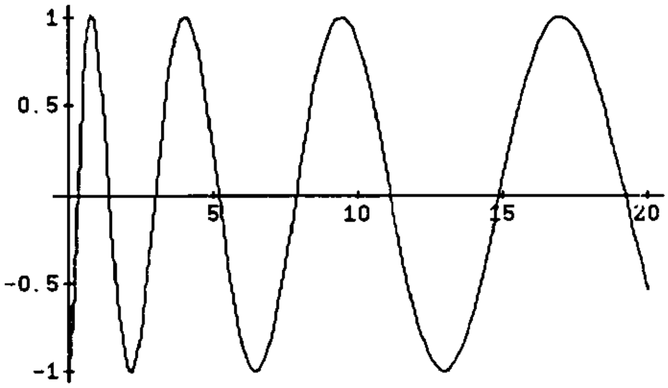


Fig. 18. $y_1(\pi)$ versus separation constant c, out-of-plane (P_0).

TABLE VII
Intervals of c for unstable solutions (out-of-plane).

	c_{low}	c_{high}	$c_{hi} - c_{lo}$	c_{max}	$y_1(\pi)$	ν_i
0			0	0	−1	0
1	0.772	0.850	0.078	0.8	1.0082	0.86015
2	2.10	2.20	0.100	2.15	−1.00423	1.6331
3	4.01	4.-7	0.060	4.-4	1.00157	2.3862
4	6.44	6.48	0.040	6.46		3.1290
5	9.41	9.43	0.020	9.42	1.00016	3.8692
6	12.91	12.93	0.02	12.92	−1.00005	4.6078
7	16.952	16.961	0.009	16.91	1.00001	5.34551
8			0	21.53	−1	6.0810
9			0	26.64	1	6.8164
10			0	32.29	−0.999999	7.5517

For the in-plane function we see from Table IV that in order to have $P_i(\vartheta) > 0$ we need $c > 3.7676$ or $\nu > 2.290$. For smaller values of c one solution is likely to be exponentially unstable.

The bold values in Table VIII are unstable. The function is given in Figure 17. As $y_1(0) = 629.3$, $y_1(15) = -0.9836$, $y_1(21) = 0.014$ the two rigid modes and the first two flexible modes are unstable for a tether without tip masses. The result for the rigid modes is related to the assumptions about the motion of the COM. The instability in rotation ($c = m = 1$) is linear in time as there is one periodic solution. There is no 'stiffness' or restoring force for the particular rate of rotation that was chosen. The tether can a priori revolve at any rate. The interpretation is that these two instabilities are related to the problem formulation and that they

TABLE VIII
$y_1(\pi)$ values (in plane).

c	+0.00	+0.20	+0.40	+0.60	+0.80
1	1	**8.44**	**−9.63**	**−7.20**	**−3.79**
2	−0.74	**1.37**	**2.47**	**2.70**	**2.32**
3	**1.58**	0.72	−0.09	−0.74	**−1.16**
4	**−1.35**	**−1.33**	**−1.13**	−0.8	−0.41
5	−0.01	0.37	0.68	0.91	**1.04**
6	**1.07**	**1.01**	0.87	0.66	0.42
7	0.15	−0.13	−0.38	−0.61	−0.79
8	−0.92	−0.997	**−0.013**	−0.974	−0.883
9	−0.748	−0.578	−0.383	−0.0172	0.044
10	0.254				

TABLE IX
Intervals of c for unstable solutions (in-plane).

	c_{low}	c_{high}	$c_{hi} - c_{lo}$	c_{max}	$y_1(\pi)$	ν_t
0	0	1	1	0	629.3	
1	1.03	1.9	0.87	1.4	−9.63	
2	2.1	3.1	1	2.6	2.70	
3	3.7	4.45	0.75	4.0	−1.35	2.372
4	5.77	6.2	0.48	6.9	1.07	3
5	8.22	8.50	0.28	8.36	−1.01	3.619
6	11.21	11.38	0.17	11.29	1.00295	4.278
7	14.75	14.82	0.07	14.78	−1.00064	4.960
8	18.79	18.82	0.03	18.805	1.00014	5.653
9	23.36	23.38	0.02	23.37	−1.00003	6.355
10				28.48	1.00001	7.064
11				34.13	−1	7.777

are not dangerous. These modes can not be influenced by the mass parameters. *The instability of the first two deformation modes shows that a tether without end masses does not retain its straight line configuration while revolving.*

Tabel IX gives the distribution of the in-plane instability intervals. As expected instability is the rule rather than the exception for $c < 3.76$. When c increases further the instability intervals seem again to contract to points where the two periodic solutions coexist. A practical stability (= slow enough instability) is achieved from $c = 23.37$ or $\nu = 6.355$ onwards.

8. Analytic Approximation of the Critical Points

To apply the results of Appendix A (Equation 12) we evaluated:

$$\varphi_\pi = \int_0^\pi \sqrt{A_n}\, dt \qquad \Rightarrow \varphi_\pi = 6.05625 \tag{45}$$

$$\alpha_\pi = \int_0^\pi \frac{\Pi_{i0}}{2\sqrt{A_n}}\, dt - \frac{1}{32}\int_0^\pi \frac{A_n'^{\,2}}{a_n^{5/2}}\, dt \Rightarrow \alpha_{\pi 0} = 0.808742 \tag{46}$$

$$\alpha_{\pi i} = -4.71397\,. \tag{47}$$

The asymptotic separation between the critical values *for in-plane and out-of-plane* is

$$\Delta\sqrt{c} = \frac{\pi}{\varphi_\pi} = 0.5186 \Rightarrow \Delta\nu = \sqrt{2}\,\Delta\sqrt{c} = 0.7335\,.$$

This value is close to the observed 0.735 for the out-of-plane critical values. The last two in-plane differences from Table X are 0.709 and 0.713. The in-plane difference is not yet stabilised but going to the predicted value. An approximate formula for the critical values (Equation 14, Appendix A) can be written as:

$$\boxed{\sqrt{c_n} = n\,\Delta\sqrt{c}\left\{\frac{1}{2} + \sqrt{\frac{1}{4} - \frac{\alpha_\pi}{\pi\Delta\sqrt{c}}\,\frac{1}{n^2}}\right\}} \tag{48}$$

Table X gives a comparison of Equation 48 with the c_{max} values.

$$\text{Out-of-plane} \Rightarrow \sqrt{c_n} = 0.519n + \left\{0.5 + \sqrt{0.25 - \frac{0.4963}{n^2}}\right\}$$

$$\text{In-plane} \qquad \Rightarrow \sqrt{c_n} = 0.519n + \left\{0.5 + \sqrt{0.25 + \frac{2.8928}{n^2}}\right\}.$$

The precision for the extrema of $\sqrt{c_{max}}$ is remarkably good, also for the lower values of c, where the instability interval is not negligible. Notice the shift in n between the in-plane and out-of-plane values.

TABLE X
Numerical and analytical critical values.

Out-of-plane		In-plane	
$\sqrt{c_{max}}$ (num)	$\sqrt{c_n}$	$\sqrt{c_{max}}$ (num)	$\sqrt{c_n}$
0.894	0.8868 ($n = 2$)	1.183	1.179 ($n = 1$)
1.466	1.465	1.612	1.543
2.010	2.008	2.000	1.955
2.542	2.5409	2.449	2.400
3.069	3.0687	2.891	2.867
3.594	3.596	3.360	3.347
4.112	4.119	3.844	3.836
4.640	4.642	4.336	4.332
5.161	5.164	4.834	4.832
5.682	5.685	5.337	5.336
		5.842	5.842

9. Adjustment of the End Masses

The in-plane instability for a revolving tether can be removed by adding appropriate end-masses. It is sufficient to consider equal end masses. From Tables VII and IX we have the minimal values of $\sqrt{c}$ to give the critical points the character of periodic solutions. The in-plane condition dominates $\sqrt{c} > \text{MAX}\,\{3.594, 4.834\}$. This can be translated in a $\Delta\nu \sim \nu_2$ value $\Delta\nu = \sqrt{2}\,4.834 = 6.836$ which in turn puts a lower limit on the ratio of the end mass over the distributed mass:

$$s_n = \cos\left[\frac{\pi}{2}\left(1 - \frac{1}{\Delta\nu}\right)\right] = 0.2277$$

$$r_n = \frac{1}{2s_n^2} - \frac{1}{2} = 9.14\,.$$

For a tether of 200 kg this means end masses of 1827.6 kg. Smaller end masses can be used by checking that the lowest value $\sqrt{c}$, corresponding to ν_2, is not critical and making $\Delta\nu$ a multiple of $\Delta\sqrt{c}$. Table XI shows that an end mass of 1.8 times the tether mass gives a $\sqrt{c_2}$ value of 3.387 which is in between 3.069 and 3.594 (out-of-plane) and 3.360 and 3.844 (in-plane). The analytical approximations are sufficient for the out-of-plane check. In-plane, this value is just above the unstable interval $3.348 - 3.370$. For mass ratios which give a multiple of $D\sqrt{c}$ the $\sqrt{c}$ value from m_2 is systematically close to a critical in-plane value. It is better to select the mass ratio such that $\sqrt{c_2}$ and $\sqrt{c_3}$ move away from a critical point. The general conclusion is that *end masses are needed to avoid in-plane instabilities*. Depending

TABLE XI
Adjusted end masses.

m	$\Delta\mu$	s_{ni}	$m_i/\sigma l$	μ_i $(i > 1)$	c_i	$\sqrt{c_i}$
6	4.401	0.3494	1.798	4.31541	11.469	3.387
				8.52141	40.568	6.369
				12.8518	89.010	9.434
				17.2171	156.823	12.523
7	5.134	0.3010	2.510	4.9922	14.957	3.867
				9.557	50.447	7.103
				15.0296	120.459	10.975
				20.1336	212.748	14.586
8	5.868	0.2645	3.324	5.68265	18.988	4.357
				11.399	70.668	8.406
				17.2142	156.771	12.521
				23.0557	277.310	16.653
9	6.601	0.2357	4.251	6.38254	23.560	4.854
				12.8483	88.964	9.432
				19.4032	197.944	14.069
				25.9817	350.515	18.722
10	7.335	0.2125	5.287	7.08885	28.670	5.354
				14.301	109.410	10.460
				21.5945	243.958	15.619
				28.9091	432.323	20.792

on the mission and the tether properties, end masses that assure in-plane stability are readily calculated.

Appendix A: Hill's Equation

$$y'' + P(t;\, \lambda)y = 0\,. \tag{A1}$$

The study of the linear differential Equation A1 with $P(t;\, \lambda) > 0$ and periodic is a specialised subject. $P(t)$ depends on a parameter λ, and has period T. From $P > 0$ it follows that P can be considered as even when the origin is chosen in an appropriate way (translation).

If P were a constant the solutions would be the harmonic functions with period $2\pi/\sqrt{P}$ or circular frequency $\omega = \sqrt{P}$. These solutions are always periodic and bounded. The surprising fact is that for $P(t)$ satisfying the conditions mentioned above, the solutions can be unbounded for particular values of λ. This happens also when the variation of P over a period is small compared to its average value.

In general the solutions to A1 are not periodic. The conditions under which one of the two solutions have period T or $2T$ is known as Floquet theory. Summaries of Floquet theory are given in Hochstadt (1975) and Nayfeh and Mook (1979). In most of the cases when there is a periodic solution (period T or $2T$) the other solution is unbounded. This is proved for the Mathieu equation ($P = a + b \cos 2t$) by Ince (1956).

From the analogy with a harmonic oscillator we expect that when $\overline{\sqrt{P}}$* *is a multiple of* T the oscillator sees an excitation (parametric) in phase with its resonance frequency. This can be thought of as an internal resonance condition. The exact dependency of P on λ seems to play a role to a lesser extent. Magnus (1966) obtained strong results when considering $P(t) = \lambda + Q(t)$ where $Q(t)$ is even periodic independent of λ. In this case $\overline{\sqrt{P}}$ goes to $\sqrt{\lambda}$ for λ large and the resonance condition becomes $\sqrt{\lambda} = n\,T/\pi$. The results of Magnus are not directly applicable to $P_{i,0}(t)$ which have a structure:

$$P(t)_{i,0} = \Pi_{i,0}(t) + \lambda\, A_n(t)\,. \tag{A2}$$

The variation of the functions $P_{i,0}(t)$ depends on the parameter. $A_n(t)$ is the same for the in-plane as the out-of-plane equation. It will be shown that a similar result remains valid. Assume an even periodic solution of the type:

$$\boxed{y(t) = A(t)\cos[F(t)]} \tag{A3}$$

where $A(t)$ is even periodic. Substituting Equation A3 and its second derivative in Equation A1 gives:

$$[A'' - A\,F'^2 + AP]\cos F - [2A'\,F' + A\,F'']\sin F = 0\,. \tag{A4}$$

For Equation A3 to be a solution of Equation A1, the coefficients of cos F and sin F must vanish. The condition that the coefficient of sin F vanishes gives the following relation between A and F with C a constant:

$$A = C/\sqrt{F'}\,. \tag{A5}$$

Using Equation A5 in the coefficient of cos F of Equation A4 gives:

$$P = F'^2 - \frac{1}{4}\left[\frac{F''}{F'}\right]^2 + \frac{1}{2}\left[\frac{F''}{F'}\right]'\,. \tag{A6}$$

Equation A6 is a complicated nonlinear differential equation for F. It can also be considered as a representation formula for P. Choosing:

* Average values of $X(t)$ over a period T are denoted $\bar{X}$.

$$F = \varphi\sqrt{\lambda} + \frac{\alpha}{\sqrt{\lambda}} + \frac{\beta}{\sqrt{\lambda^3}} \tag{A7}$$

with α, β, φ unknown functions (period T) the derivatives appearing in Equation A6 can be computed. By collecting the terms according to their power in λ, neglecting higher order terms (λ^{-n}, $n > 1$) and comparing the result with Equation A2, relations defining α, β, φ are obtained. Some intermediate results:

$$F'^2 = 2\alpha'\varphi' + \lambda\varphi'^2 + [\alpha'^2 + 2\varphi'\beta']\frac{1}{\lambda}$$

$$\left[\frac{F''}{F'}\right]^2 = \left[\frac{\phi''}{\phi'}\right]^2 + 2\left[\frac{\varphi''}{\varphi'}\right]^2\left[\frac{\alpha''}{\varphi''} - \frac{\alpha'}{\varphi'}\right]\frac{1}{\lambda}$$

$$\left[\frac{F''}{F'}\right]' = \left[\frac{\varphi''}{\varphi'}\right]' + \left\{\left[\frac{\varphi''}{\varphi'}\right]'\left[\frac{\alpha''}{\varphi''} - \frac{\alpha'}{\varphi'}\right] + \left[\frac{\varphi''}{\varphi'}\right]\left[\frac{\alpha''}{\varphi''} - \frac{\alpha'}{\varphi'}\right]'\right\}\frac{1}{\lambda}\,.$$

Identifying the powers in λ with Equation A2 gives:

$$\varphi'^2 = A_n \Rightarrow \varphi(t) = \int_0^t \sqrt{A_n}\,dt \tag{A8}$$

$$\alpha' = \frac{\Pi_{i0}}{2\varphi'} + \left\{\frac{1}{4}\left[\frac{\varphi''}{\varphi'}\right]^2 - \frac{1}{2}\left[\frac{\varphi''}{\varphi'}\right]'\right\}\frac{1}{2\varphi'} \tag{A9}$$

$$\beta' = -\frac{1}{2\varphi'}\left\{\alpha'^2 + \frac{1}{2}\left[\frac{\alpha''}{\varphi'}\right]'\right\}. \tag{A10}$$

The definition of Φ requires $A_n > 0$. The coefficient is the same for the in-plane and the out-of-plane equation. It is computed by numerical integration. Using the result for φ, α can be rewritten:

$$\alpha(t) = \int_0^t \frac{\Pi_{i0}}{2\sqrt{A_n}}\,dt - \frac{1}{32}\int_0^t \frac{A_n'^{\,2}}{A_n^{5/2}}\,dt\,. \tag{A11}$$

After computing:

$$\varphi_T = \int_0^T \sqrt{A_n}\,dt \qquad \alpha_T = \int_0^T \frac{\Pi_{i0}}{2\sqrt{A_n}}\,dt - \frac{1}{32}\int_0^T \frac{A_n'^{\,2}}{A_n^{5/2}}\,dt \tag{A12}$$

and making λ large enough to neglect the term in β:

$$F(t+T) = F(t) + \varphi_T \sqrt{\lambda} + \frac{\alpha_T}{\sqrt{\lambda}} \,. \tag{A13}$$

This is a periodic solution to Equation A1 provided the change of phase $F(t + T) - F(t) = 2\pi$. The same analysis with an odd periodic function, by choosing $\sin F$ in Equation A3, would give the same result. This means that the unstable intervals reduce to points when λ increases. The analysis does not indicate if this happens for a finite λ or not. Ultimately, these values will settle on:

$$\sqrt{\lambda_k} = \frac{\sqrt{k^2\pi^2 - 4\varphi_T\alpha_T + k\pi}}{2\varphi_T} \tag{A14}$$

where the phase shift used is π to account for the periodic solutions $2T$. The corresponding asymptotic separation between the λ values that give periodic solutions is:

$$\Delta\sqrt{\lambda} = \frac{\pi}{\varphi_T} \,. \tag{A15}$$

When the result A14 is used as an estimate for λ_k, the k does not necessarily correspond to the true kth interval as the result obtained assumed λ large. This holds in particular for the in-plane periodic function which is not positive for small values of the separation constant.

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