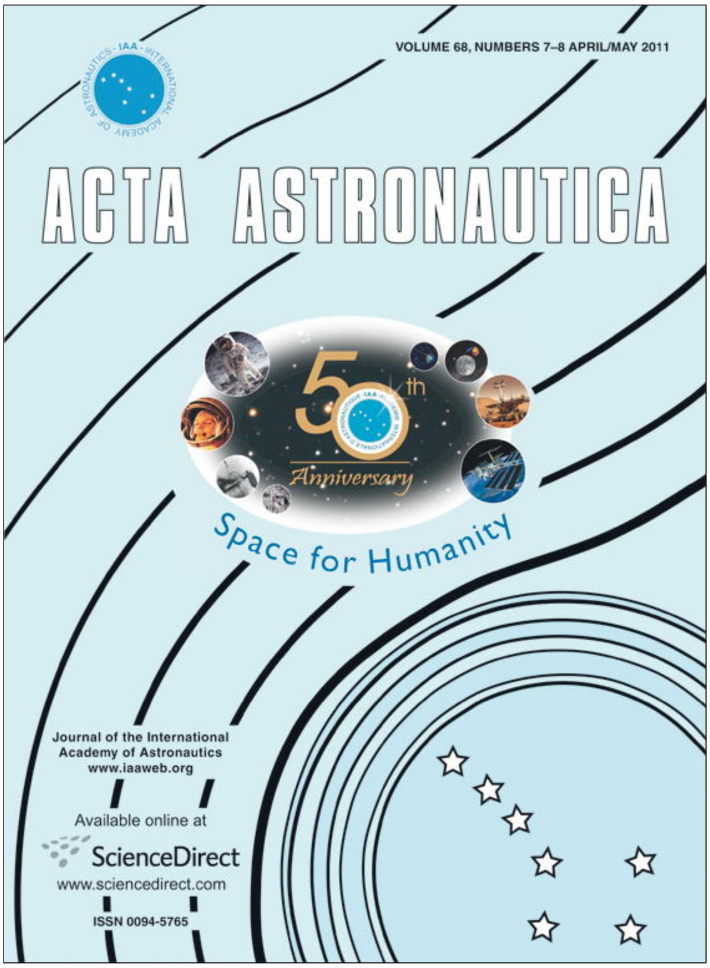
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On the stability of spinning satellites ☆

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ABSTRACT

We study the directional stability of rigid and deformable spinning satellites in terms of two attitude angles. The linearized attitude motion of a free system about an assumed uniform-spin reference solution leads to a generic MGK system when the satellite is rigid or deformable. In terms of Lyapunov's stability theory, we investigate the stability with respect to a subset of the variables. For a rigid body, the MGK system is 6-dimensional, i.e., 3 rotational and 3 translational variables. When flexible parts are present the system can have any arbitrary dimension. The 2×2 McIntyre–Myiagi stability matrix gives sufficient conditions for the attitude stability. A further development of this method has led to the Equivalent Rigid Body method. We propose an alternative practical method to establish sufficiency conditions for directional stability by using the Frobenius–Schur reduction formula. As practical applications we discuss a spinning satellite augmented with a spring–mass system and a rigid body appended with two cables and tip masses. In practice, the attitude stability must also be investigated when the spinning satellite is subject to a constant axial thrust. The generic format becomes MGKN as the thrust is a follower force. For a perfectly aligned thrust along the spin axis, Lyapunov's indirect method remains valid also when deformable parts are present. We illustrate this case with an apogee motor burn in the presence of slag. When the thrust is not on the spin axis or not pointing parallel to the spin axis, the uniform-spin reference motion does not exist and none of the previous methods is applicable. In this case, the linearization may be performed about the initial state. Even when the linearized system has bounded solutions, the non-linear system can be unstable in general. We illustrate this situation by an instability that actually happened in-flight during a station-keeping maneuver of ESA's GEOS-I satellite in 1979.

1. Introduction

Ever since the beginning of the space age, the issue of spin stability has received a great deal of attention. While Sputnik-1 rotated about its maximum axis of inertia [1], the first American satellite, Explorer-1 was designed to spin about its minimum-inertia axis. Its subsequent 'flat spin' motion after only one orbit was a complete surprise to the satellite designers. The Explorer-1 experience motivated a great deal of research on the stability of spinning spacecraft. The history of the understanding of the spin-stabilization concept is described by Barbara and Likins [2], and Hall [3]. Eventually, the maximum-inertia rule for ensuring spin stability under energy dissipation became well established and many commercial and scientific satellites have utilized this technique successfully.

In this paper we use the term 'attitude' for the 'stabilized direction' which refers to the pointing direction of the spin axis. Hughes [4] proposes the alternative term

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'directional stability'. Both of these designations are commonly used by researchers in the field of spinning satellites. When dealing with three-axis stabilized satellites, attitude refers to the orientation of an orthogonal triad of unit vectors or reference axes (see [5, Chapter 12]). The attitude of a spinning satellite, however, consists of only two parameters, i.e., the right ascension and declination angles that define the unit vector of the stabilized direction. For the sake of completeness, we mention that the spin-axis attitude may be time-varying because of nutation or due to spin-axis tilt (i.e., dynamic imbalance). In this case, only the direction of the angular momentum is constant and the attitude refers to the fixed direction of the angular momentum.

Ideally, we would like to study the spin-stability issue with the help of Lyapunov's direct method, which is applicable to the general class of non-linear systems described by $\dot{x} = f(x,t)$. This method offers a statement on stability without having to integrate the equations of motion. However, it does require an exact reference solution of the non-linear system as well as a sign-definite function which always follows from the first integrals of the system.

Lyapunov's original work was published in English in 1992 on the occasion of its 100th birthday in [6]. The method can also be found in many textbooks, for instance in [7]. Schaub and Junkins [8] formulate the Lyapunov method in the framework of control theory.

For a summary of Lyapunov's work we highly recommend the forty pages in Appendix A of Hughes [4]. He describes in detail the transition from the general Lyapunov theory to the attitude stability of spinning satellites addressed here. In spite of its generality, Lyapunov's direct method is not immediately applicable to a spinning satellite, rigid or flexible. This is because we are not interested in the stability of the full state vector x but only in the stability of the two attitude parameters. A completely free spinning satellite system is always unstable in the sense of Lyapunov.

For our problem of attitude stability, the reference motion of the non-linear system is a pure spin about a principal axis of the system. When the system is modeled as a rigid spinner, this motion is possible for any value of the spin rate. The presence of flexible appendages introduces additional deformation variables. A pure spin motion is still possible in terms of a rotating equilibrium configuration, called a 'rigidified' system, which is obvious when the system has sufficient symmetry. In general, however, the location of the Center of Mass (CoM) and the directions of the principal axes of a 'rigidified system' must be found by solving a two-point boundary value problem.

Significant progress in the investigations of attitude stability was achieved by McIntyre and Miyagi [9]. They established a 2×2 stability matrix by applying Lyapunov's direct method to a general set of linearized equations for a rigid spinner augmented with flexible parts. Their method produces a sufficient condition for attitude directional stability. Because they use Lyapunov's direct method, a derivation of the linearized equations of motion is not needed. This is an essential benefit because this is a tedious task even when only simple deformable parts are studied (e.g., pendulum, partially restrained particle). An important application of their method is the influence of fluids on the directional stability. Their interpretation that the stability results can be understood as a consequence of a shift in balance evolved further into the concept that is best known as the 'Equivalent Rigid Body', see Damilano [10]. The most important applications of both methods are for liquids.

In this paper we start with the linearized equations which have the well-known MGK structure (see [4,11,12]). When viscous damping is added to the system, the generic form of the linearized equations will take the MGDK form. We just mention here that, in contrast to a generic MK system, the necessary and sufficient stability conditions cannot be formulated in terms of matrix properties. The only attractive result is a sufficiency criterion for instability obtained by Hagedorn [13].

In practice, we often use Lyapunov's indirect method (i.e., the eigenvalue problem) for a general MGK system because the investigation of attitude stability is only a prerequisite for a full dynamic analysis. We show a systematic reduction on the attitude variables using the Frobenius–Schur formula for partitioned matrices. The resulting 2×2 matrix is precisely the stability matrix obtained by McIntyre and Myagi [9]. For illustration, we present two fairly straightforward examples of MGK systems with nonetheless considerable practical relevance. The first one considers a rigid body appended with two cables and the second one deals with a rigid body augmented with a mass-spring system. This model serves as a starting point for describing a nutation damper. The use of generalized coordinates provides a useful check because the linearized equations must have the correct generic format.

In practical applications we have also to deal with stability under thrusting. Because the thrust is a follower force, the generic format of the problem becomes an MGKN system. We analyze a rigid body augmented by a mass particle, which is nominally located on the spin axis but is free to move around. This is a simplification of the model used by Mingori and Yam [14] to investigate the instability of the PAM-D upper stage (i.e., the slag model).

When the thrust is not on or along the spin axis, an exact solution of the full non-linear equations is not known even for a single rigid body and Lyapunov's direct method is not applicable. In this case, we may perform a linearization based on an *initial state* whose validity is limited in time. The stability of these linear equations does definitely not imply the stability of the non-linear system. We illustrate this by an example that occurred in practice on ESA's GEOS-I satellite in 1979 when a station-keeping maneuver was performed using a single axial attitude thruster pointing in the direction of the desired delta-V (see [15]). Although the linearized equations have only bounded solutions, a higher-order perturbation analysis reveals an *instability* that causes a rapid despin of the satellite in the case of GEOS see also [16]. This prediction was verified by an experiment when GEOS-I had reached its end of life. When also cables are present, as was the case in GEOS-I, the stability result remains essentially the same.

2. MGK system for a free spinning satellite

2.1. Pure-spin reference motion

The attitude motion of a spinning system, rigid or deformable, can be described by a non-linear system of equations. For a free rigid body, this system consists of the Euler equations completed by the non-linear differential equations relating the angular velocity to the chosen attitude variables. The deformable parts require supplementary equations. We assume that the non-linear equations possess the uniform pure-spin solution ω_1=ω_2=0, ω_3=Ω=constant. In this motion, the system rotates uniformly as a rigid body with static deformations of the flexible and/or deformable parts. This configuration is called 'rigidified configuration' or 'equilibrium configuration' and its mass properties must be identified a priori. This pure-spin solution is the reference solution for all applications considered here. The attitude is studied in the 'equilibrium frame' (in short, E-frame) with its origin at the CoM of the rigidified configuration. The axes of the E-frame are aligned with the principal axes of the rigidified configuration. The E-frame rotates at a constant rate Ω about the z-axis which coincides with the angular momentum vector H=$C\Omega$ where C is the moment of inertia of the rigidified configuration about the z-axis.

When the system has sufficient symmetry it is straightforward to determine the mass properties of the equilibrium configuration. In general, however, this may involve solving a non-linear equilibrium problem to obtain the location of the CoM, the total principal inertias, and the reference values of the deformation variables when all deformable parts of the system are at their equilibrium locations.

2.2. Attitude angles

The spin-axis attitude refers to the orientation of the z-axis of the rigidified configuration. The small deviations of the attitude within the E-frame can be represented by a linear model in terms of the three small rotation angles $\{\theta_1, \theta_2, \theta_3\}$, see Fig. 1. This set of angles forms a {1-2-3} Euler sequence of small rotations and is also known as the

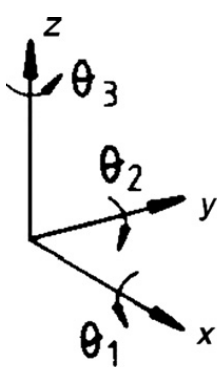

Fig. 1. Tait–Bryan angle rotations.

Tait–Bryan angles. The spin-axis attitude stability is described by the two angles $\{\theta_1, \theta_2\}$ which are called the 'attitude variables'.

Because the angles $\{\theta_1, \theta_2, \theta_3\}$ are first-order quantities, their sequence has no relevance. The rotation matrix R from the E-frame to the instantaneous body frame can be written as

$$R=\begin{bmatrix}1 & \theta_3 & -\theta_2\\ -\theta_3 & 1 & \theta_1\\ \theta_2 & -\theta & \theta\end{bmatrix} \tag{1}$$

2.3. MGK system of equations

Using Eq. (1) we obtain a set of second-order differential equations for the rotational motion in terms of the state vector q=$(\theta_1, \theta_2, \theta_3, q_d)^{\mathrm{T}}$, see [3,4,11]

$$M\ddot{q}+G\dot{q}+Kq=0 \tag{2}$$

where q_d is the set of deformation variables which vanishes in the rigidified configuration. Eq. (2) is the standard format of a linear conservative MGK system with matrices M=$M^{\mathrm{T}}>0$, G=$-G^{\mathrm{T}}$, and K=K^{T}. When the deformations couple to the translations of the rigid body, we have to include the 3 translational degrees of freedom of a rigid body and the state vector becomes: $(x, y, z, \theta_1, \theta_2, \theta_3\ q_d)^{\mathrm{T}}$. In general, the dimension of the matrices may take an arbitrary value n as the deformation variables are included in the state vector.

We also introduce the dynamic stiffness matrix Z

$$Zq=0 \quad \text{with} \quad Z(p)=Mp^2+Gp+K \tag{3}$$

where a time dependency e^{pt} with $p\sigma$=$+i\Omega$ is introduced in Eq.(2). The corresponding characteristic equation is

$$\det[Z(p)]=0 \tag{4}$$

This is a polynomial equation in p of degree $2n$ where n is the order of the matrices and contains only even powers of p. Because the matrices M, G, and K are real, the complex roots of Eq. (4) occur in conjugate pairs. When all roots are on the imaginary axis, i.e., p_j=$\pm i\omega_j$, the solutions combine to harmonic functions and the system is oscillatory stable (grenzstabil). The eigenvectors corresponding to $\pm i\omega_j$ are conjugate complex. The multiplicity of an eigenvalue is not a problem provided that the number of independent eigenvectors equals the multiplicity.

As an example, for a rigid body these 3×3 matrices are

$$M=\begin{bmatrix}A & 0 & 0\\ 0 & B & 0\\ 0 & 0 & C\end{bmatrix};\quad G=\Omega\begin{bmatrix}0 & -\alpha & 0\\ \alpha & 0 & 0\\ 0 & 0 & 0\end{bmatrix};$$

$$K=\Omega^2\begin{bmatrix}C-B & 0 & 0\\ 0 & C-A & 0\\ 0 & 0 & 0\end{bmatrix} \tag{5a−c}$$

where α is defined as α=$A+B-C$ and A, B, C are the principal inertia's of the body which equal the rigidified system inertias.

Eq. (2) is quite general and includes singular cases. This can be illustrated by means of Eqs. (5), which indicate that the variable θ_3 is completely decoupled from the two attitude variables θ_1 and θ_2. This causes an instability called a 'rigid mode'. The presence of θ_3 makes the K matrix semi-definite positive. The physical interpretation is that the rigid body may spin about its z-axis at any spin rate of the reference solution. Therefore, if it spins at $\Omega' \neq \Omega$, the angle θ_3 would change linearly in time, which corresponds to an *instability* in the sense of Lyapunov.

For the complete non-linear system, the Lyapunov functions required for the direct method are not known, not even for a single rigid body. They are only available for the Euler equations in the angular velocities [4,7]. So, we have always to deal with Lyapunov's indirect method as the MGK systems follow from a linearization about a reference solution. If the MGK formulation represents truly the physical system, its stability may be investigated with Lyapunov's direct method. In fact, for such systems there is a systematic method to construct quadratic Lyapunov functions by solving the Lyapunov matrix equation, see [7,11]. In the case of a MGK(N) system, Lyapunov's indirect method is not able to confirm the stability of the non-linear system. Only when the linearized system is asymptotically stable (which implies damping) it is possible to confirm the stability of the reference solution of the non-linear system (see [4,Theorem A15]).

2.4. Attitude stability regions

As an application of Lyapunov's indirect method, we discuss the characteristic equation corresponding to Eqs. (3) and (5), namely the rotational degrees of freedom of a rigid body. The characteristic equation (see Eq. (4)) leads to the sixth-order polynomial equation for the attitude angles

$$p^2(p^2+\Omega^2)\left(p^2+\Omega^2\frac{(C-A)(C-B)}{AB}\right)=0 \tag{6}$$

The roots $p=0$ correspond to the rigid mode of the θ_3 angle discussed above. The roots of the last part of Eq. (6) define the nutation frequency. They are on the imaginary axis only when the conditions $C>A$, B or $C<A$, B are satisfied.

Fig. 2 shows a diagram with axes $x=A/B$ and $y=C/B$. All possible rigid bodies are in an infinite strip along the first diagonal (A, B, $C>0$) and each is at most the sum of the other two. The region $K>0$ characterizes a rotation about the maximum axis. The region $K<0$ is a rotation about the minimum axis and is also stable. In the two remaining regions K is not sign-definite. (A different presentation is given by the so-called Smelt-plane, see Hall [3] and Hughes [4], where the axes are $y=(B-A)/C$, $x=(B-C)/A$ and all possible bodies are contained within the rectangle ± 1, ± 1.)

The roots of the middle part of Eq. (6), i.e., $p=\pm i\Omega$, are a consequence of the choice of variables θ_1, θ_2. The interpretation of these roots is that the angular momentum can spin (stably) about a direction that differs from the z-axis of the E-frame which is its original direction. This change of the direction of angular momentum is discussed at length by Barbara and Likins [2] and Hughes [4]. Strictly speaking, this mode represents an attitude instability but it shows up as a stable mode. In the common meaning of 'attitude stability', this mode may be disregarded.

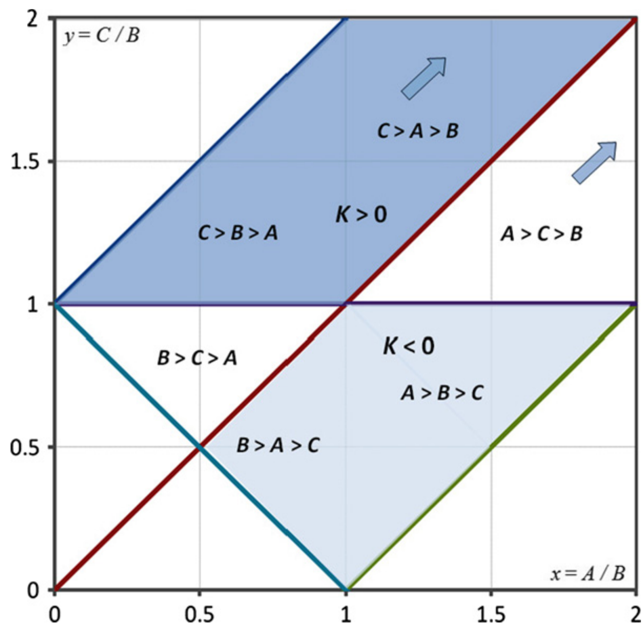

Fig. 2. Stability regions.

2.5. Influence of damping

In the case when the addition of damping makes the linearized system asymptotically stable, the non-linear system will also be asymptotically stable, see [4, Theorem A15]. To illustrate this theorem we include viscous damping by adding terms $d_1\dot{\theta}_1$, and $d_2\dot{\theta}_2$ with d_1, $d_2 \geq 0$ to Eqs. (5).

When d_1, $d_2>0$, the matrix $D=D^{\mathrm{T}}=\mathrm{diag}\{d_1, d_2\}$ is positive definite. When adding this matrix D to the matrices in Eqs. (2) we obtain a MGDK system. The theorem of Thomson–Tait states "in the case when $D>0$ and $\det(K)\neq 0$, the stability of the MDGK system follows from the stability of the MK system", see [11].

When assuming $D>0$, the Thomson–Tait conditions are satisfied for a satellite spinning about its maximum inertia as well as about its minimum-inertia axis. For the maximum axis we have $K>0$ which is a sufficient condition for stability and the stability is guaranteed. For the minimum-inertia spin axis, the K matrix is negative definite and we cannot conclude immediately. The assumption $D>0$ is very restrictive and can be replaced by: "a D matrix such that the damping is pervasive", see [11]. This means that the damping is such that it affects all the modes of the system.

As an example, we take the D matrix as diag$\{d_1, 0\}$. When considering a body with inertias A, B, $C=\{50, 75, 100\}$, the characteristic equation for the variables θ_1, θ_2 is given by the fourth-order polynomial (after normalization for $\Omega=1$)

$$(p^2+1)(p^2+1/3)+pd_1(3p^2+2)/150=0 \tag{7}$$

Fig. 3a and b show the evolution of the four roots as functions of d_1. When the satellite is spinning about its maximum-inertia axis the nutation frequency is $\Omega/\sqrt{3}$

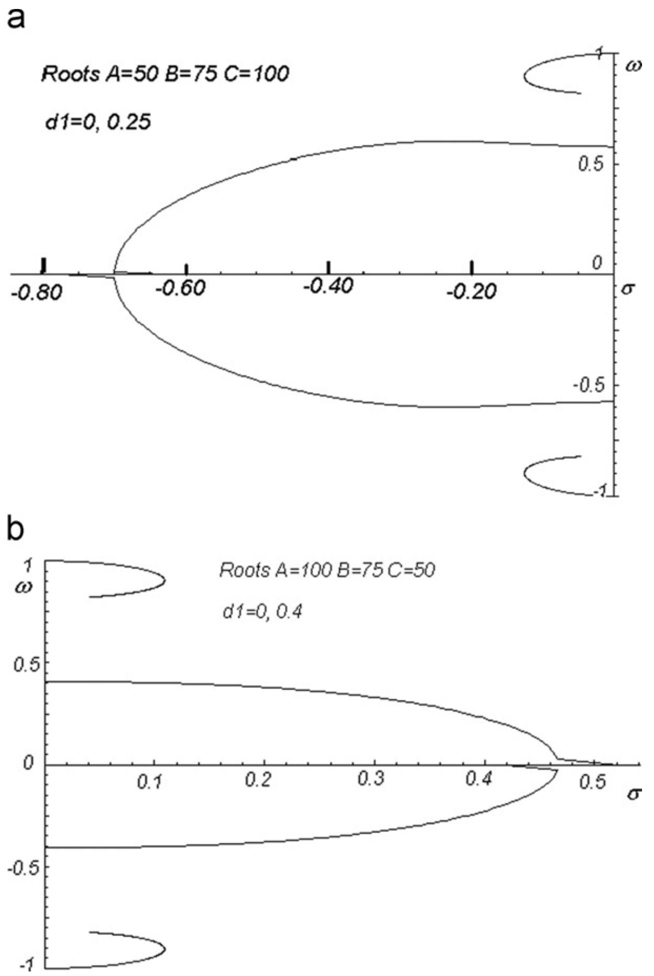

Fig. 3. (a) Stabilization about maximum-inertia axis. (b) Stabilization about minimum-inertia axis.

(see Fig. 3a). Both frequencies (of the four roots) have negative real parts as expected. The horizontal departure of the spin frequency on the imaginary axis implies that, up to second order, the spin axis returns to its initial direction.

Fig. 3b shows the evolution of the roots for the case when the same body is initially spinning about its minimum axis with the body inertias A, B, C={100, 75, 50} and the determinant equation

$$(p^2+1)(p^2+1/6)+pd_1(3p^2-2)/300=0 \tag{8}$$

In this case, the nutation frequency is $\Omega/\sqrt{6}$ and both frequencies have now positive real parts and the body will proceed to go into a flat spin. Thus, we have now established numerically the stability properties of the attitude of spinning rigid bodies, including damping, by means of Lyapunov's indirect method.

3. Attitude stability of a free spinning satellite

In practice, we are only interested in the variables θ_1 and θ_2 that define the 'attitude stability', i.e., stability in terms of a subset of the system variables. When the stability theorems are applied to the full MGK system, the results are of course also valid for the attitude. We present a method that provides stability results for the spin-axis attitude variables θ_1 and θ_2 while using only the 2×2 matrix. In the rigid-body model that was presented in Eqs. (3) the reduction to the two attitude variables is straightforward because the variable θ_3 as well as the translation variables are completely decoupled from θ_1 and θ_2. It suffices to simply ignore their equations of motion to arrive at the 2×2 equations for θ_1 and θ_2.

3.1. Frobenius–Schur reduction

In order to establish the general procedure for this reduction, we start from the dynamic stiffness matrix $Z(p)$ given in Eq. (3). We recall that Z (p=0) is equal to the matrix K. Eqs. (5b,c) indicate that the matrices G and K can be highly singular in practical applications. When the matrix $K>0$ the system is called 'statically stable', see [4,11].

In order to separate the attitude variables from the deformation variables we partition the vector q in two parts, i.e., $q^T=[q_1^T, q_2^T]$, so we write Eq. (3) as

$$Z_{11}q_1+Z_{12}q_2=0; \quad Z_{21}q_1+Z_{22}q_2=0 \tag{9a, b}$$

Now we can eliminate q_2 by writing

$$q_2=-Z_{22}^{1}Z_{21}q_1 \Rightarrow [Z_{11}-Z_{12}Z_{22}^{1}Z_{21}]q_1=0 \tag{10a, b}$$

After substituting $q_1=(\theta_1,\ \theta_2)^T$ in Eq. (10b) we can formulate the condition for 'static stability' in terms of the 2×2 matrix Z_R (p) as

$$Z_R=[Z_{11}-Z_{12}Z_{22}^{1}Z_{21}] \Rightarrow \tag{11a}$$

$$Z_R(0)=K_R=[K_{11}-K_{12}K_{22}^{1}K_{21}]>0 \tag{11b}$$

The matrix Z_R is known as the 'Frobenius–Schur reduction formula' for partitioned matrices, see [17]. It assumes that no external forces are acting on the eliminated variables.

The expression in Eq. (11b) contains the inverse of the matrix K_{22} which may in fact not exist in a specific practical application. When this happens, the product $K_{12}K_{22}^{-1}K_{21}$ remains well defined and becomes 0 when the equations represent a physical system. The final result K_R=K_{11} is then identical to the matrix which would turn up by simply deleting the equations belonging to the eliminated variable(s). Instead of applying the Frobenius–Schur reduction formula to the Z matrix, we may also use it directly on the K matrix. This is apparent because of the resemblance of the matrices of Eqs. (10b) and (11b).

When applying the above reduction procedure to the matrix in Eq. (5c) we find the static stability condition

$$K_R=\Omega^2\begin{bmatrix} C-B & 0 \\ 0 & C-A \end{bmatrix}>0 \tag{12}$$

This means that the well-known maximum-inertia condition represents a *sufficient* condition for attitude stability. Thus, the matrix K_R, which follows from the linearized equations of rigid-body motion in MGK format, provides the stability condition for the 2×2 system of the attitude variables θ_1 and θ_2.

3.2. McIntyre–Miyagi model

The pioneering paper by McIntyre and Myagi [9], or M&M in short, establishes a 2×2 stability matrix by applying Lyapunov's direct method to a general linearized system with flexible parts.

The stability theory of M&M starts with the diagonal matrix K_0=diag{$C-B$, $C-A$}. The moments of inertia A, B, C represent the 'rigidified' *principal* system inertias. The matrix K_0 can thus be seen as the counterpart of the matrix K_{11} in K_R of Eq. (11b).

We introduce now the M&M notations that allow us to compare their results with our matrix K_R in Eq. (11b). Their derivation considers the 3×3 system inertia matrix J=[J_{ij}] as a function of the deformation variables q_d. A Taylor expansion of the system inertia matrix in terms of q_d obviously has diag{A, B, C} as the leading term. In general, the higher-order terms of the matrix elements $J_{ij}(q_d)$ are non-zero and we introduce the derivatives of the inertia terms as in M&M, Eqs. (20)

$$a_i = \frac{\partial J_{13}}{\partial q_{d,i}};\quad b_i = \frac{\partial J_{23}}{\partial q_{d,i}} \tag{13a, b}$$

Their derivation shows that the only second derivative needed is the one of the spin moment of inertia J_{33}. It appears in the matrix Γ which is defined by the following quadratic form:

$$q_d^{\mathrm{T}}\Gamma q_d = J_{33}^2\Big|_{(1)}/C - J_{33}\Big|_{(2)} + 2U\Big|_{(2)}/\Omega^2 \tag{14}$$

where (1) and (2) denote the order of the derivative with respect to q_d, Ω is the nominal spin rate, and U is the *elastic* potential energy of the deformations.

With these definitions we can write the stability matrix given in M&M, Eq. (24), in the form

$$K_{M\&M} = K_0 - C_{\mathrm{d}} \tag{15a}$$

with

$$C_d = \begin{bmatrix} b^{\mathrm{T}} \\ a^{\mathrm{T}} \end{bmatrix}\Gamma^{-1}\begin{bmatrix} b \\ -a \end{bmatrix} = \begin{bmatrix} b^{\mathrm{T}}\Gamma^{-1}b & -b^{\mathrm{T}}\Gamma^{-1}a^{\mathrm{T}} \\ a^{\mathrm{T}}\Gamma^{-1}b & a^{\mathrm{T}}\Gamma^{-1}a \end{bmatrix} \tag{15b}$$

The symmetric 2×2 matrix C_d contains the correction terms that must be subtracted from the matrix K_0 in order to arrive at the M&M stability matrix $K_{M\&M}$ which has the same structure as the matrix K_R in Eq. (11b). In practice, the matrix $K_{M\&M}$ is identical to the matrix K_R as our examples in a later section will illustrate. The sensitivity coefficients defined in Eqs. (13) provide a physical interpretation in terms of a 'variation in inertias' which is not at all obvious when these terms are just seen as coefficients of the linearized equations of motion.

Furthermore, the M&M approach allows the interpretation that the only physical mechanism that can produce unstable behavior in a passively spinning flexible body is the changing balance. M&M refer to the existence of the matrix Γ^{-1} as the 'structural integrity condition'. The derivation by reduction in Eq. (11b) shows that a singularity in Γ may also be caused by a decoupling of one or more of the deformation variables from the attitude variables. In this case the corresponding components of a and b would vanish.

It is important to note that the M&M stability condition can be calculated without having to establish the equations of motion. This is because the elements of the matrix C_d represent the sensitivity coefficients of the system inertia relative to the deformations. Indeed, this interpretation represents a crucial advancement in our understanding. In particular, it allows the investigation of attitude stability when liquids represent the deformable parts, i.e., the concept of 'wobble amplification' (see [19]). In practice, it is rarely straightforward and often problematic to establish reliable models for coupling the equations of motion of fluids to a rigid body.

3.3. Equivalent rigid-body model

The M&M interpretation of the static attitude stability as the reaction of the system to an assumed shift in balance (i.e., inertia matrix) was further developed in the Equivalent Rigid Body (ERB) approach, see for instance [10]. As in the M&M approach, the ERB model starts with the 'rigidified' *principal* system inertias J_0=diag{A, B, C}.

Now we imagine that the deformable system has its spin axis about the axis n within the body frame

$$n = (\alpha,\ \beta,\ \gamma)^{\mathrm{T}} \tag{16}$$

where α and β are small angles with respect to the ideal spin axis of a 'perfect' satellite and $\gamma = \sqrt{1-\alpha^2-\beta^2}$. The inertia matrix of the deformed system changes to $J(n)$ which is close to J_0. The new spin moment of inertia I can be expressed in the components of n by means of Eq. (16).We retain only up to second-order terms in α, β so we have

$$I = n^{\mathrm{T}}[J]n = (\alpha,\beta,\gamma)\begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{12} & J_{22} & J_{23} \\ J_{13} & J_{23} & J_{33} \end{bmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \approx \tag{17a}$$

$$\approx J_{33} + 2\alpha J_{13} + 2\beta J_{23} + \\ + \alpha^2(J_{11}-J_{33}) + \beta^2(J_{22}-J_{33}) + 2\alpha\beta J_{12} \tag{17b}$$

The rotational energy E=$I\Omega^2/2$ of the deformed system can now be calculated as

$$E \approx \frac{\Omega^2}{2}\left\{ \begin{matrix} J_{33} + 2\alpha J_{13} + 2\beta J_{23} + \\ + \alpha^2(J_{11}-J_{33}) + \beta^2(J_{22}-J_{33}) + 2\alpha\beta J_{12} \end{matrix} \right\} \tag{18}$$

A well-known theorem (see [18, p. 124]) states that the condition for stability corresponds to the minimum of the rotational energy for a given angular momentum. The equilibrium position occurs when J_{13}=J_{23}=0 and the minimum is reached for $J_{33} > \max(J_{11},\ J_{22})$. This result confirms the 'maximum-inertia rule' for a rigid body.

Next, we expand the elements J_{kl} (k, l=1, 2, 3) of the matrix J in a second-order Taylor series about α=β=0

$$J_{kl} \approx J_{0,kl} + \left\{ \begin{matrix} \alpha\dfrac{\partial J_{kl}}{\partial\alpha} + \beta\dfrac{\partial J_{kl}}{\partial\beta} \\ + \dfrac{\alpha^2}{2}\dfrac{\partial^2 J_{kl}}{\partial^2\alpha} + \dfrac{\beta^2}{2}\dfrac{\partial^2 J_{kl}}{\partial^2\beta} + \alpha\beta\dfrac{\partial^2 J_{kl}}{\partial\alpha\partial\beta} \end{matrix} \right\} \tag{19}$$

After substituting these terms into Eq. (17b), we rearrange the rotational energy in Eq. (18) in first-and second-order contributions in the small parameters α and β, i.e., $E=I\Omega^2/2 \approx E_1+E_2$ with

$$E_1 = \frac{\Omega^2}{2}\{J_{0.33}+2\alpha(J_{0.13}+J^*_{13})+2\beta(J_{0.23}+J^*_{23})\} \quad (20a)$$

$$E_2 = \frac{\Omega^2}{2}\left\{\begin{array}{c}\alpha^2(J_{0.11}+J^*_{11}-J_{0.33})+2\alpha\beta(J_{0.12}+ +J^*_{12})\\ +\beta^2(J_{0.22}+J^*_{22}-J_{0.33})\end{array}\right\}$$
$$= \frac{\Omega^2}{2}[\alpha,\beta]\begin{bmatrix}A-C+J^*_{11} & J^*_{12}\\ J^*_{12} & B-C+J^*_{22}\end{bmatrix}\begin{bmatrix}\alpha\\ \beta\end{bmatrix} \quad (20b)$$

$$J^*_{13} = \frac{1}{2}\frac{\partial J_{33}}{\partial\alpha}; \quad J^*_{23} = \frac{1}{2}\frac{\partial J_{33}}{\partial\beta} \quad (20c,d)$$

$$J^*_{12} = \frac{\partial J_{13}}{\partial\beta}+\frac{\partial J_{23}}{\partial\alpha}+\frac{1}{2}\frac{\partial^2 J_{33}}{\partial\alpha\partial\beta} \quad (20e)$$

$$J^*_{11} = 2\frac{\partial J_{13}}{\partial\alpha}+\frac{1}{2}\frac{\partial^2 J_{33}}{\partial^2\alpha}; \quad J^*_{22} = 2\frac{\partial J_{23}}{\partial\beta}+\frac{1}{2}\frac{\partial^2 J_{33}}{\partial^2\beta} \quad (20f,g)$$

Eq. (20b) shows that the minimum of E is guaranteed when the matrix

$$K^*\begin{bmatrix}C-(A+J^*_{11}) & -J^*_{12}\\ -J^*_{12} & C-(B+J^*_{22})\end{bmatrix} \quad (21)$$

is positive definite. When recognizing that the angles α, β correspond to θ_2 and $-\theta_1$, respectively, we can see that this result is consistent with Eq. (12).

The ERB formulation does not express the underlying dependency on the deformation variables in explicit terms. In practice, the method is mainly used to investigate the influence of liquids on the stability. This is done numerically by solving the equilibrium problem with internal loops on the deformation variables. Analytical results in terms of the satellite's physical parameters are not provided. We also point out that both the ERB and M&M methods make use of the fact that the relevant part of the rotational energy is a quadratic form.

Damilano [10] also provides J^*_{ij} terms representing the influence of a thruster misalignment. We show in our last example below that this application of the ERB method is incorrect: a uniform spin is in general not an exact solution of the non-linear system in these circumstances.

4. Examples of attitude stability for free systems

In the applications presented here we deal only with the sufficient stability condition $K_R>0$ which has the most practical relevance. In spite of extensive theoretical research [19,20], the exact necessary and sufficient conditions for an MGK system are not known in terms of matrix properties.

4.1. Rigid body augmented by spring–mass system

As the first application, we consider a rigid body augmented with a spring–mass system in a tube. This was one of the very first models used to investigate the influence of deformable parts on the attitude stability for the use as nutation damper. We do not add

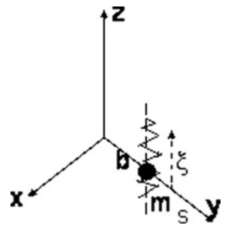

Fig. 4. Rigid body with spring–mass system.

a damping term because the stability limit is independent of damping.

We introduce a spring–mass system $\{m_s, k\}$ parallel to the z-axis and located at the position $(0, b, 0)^{\mathrm{T}}$ with a small displacement x along the x-axis, see Fig. 4. The equations for $q=(\theta_1, \theta_2, x)^{\mathrm{T}}$ are of the MGK type (see [4] and [20]) with the matrices defined by

$$M = \begin{bmatrix}A+m_s b^2 & 0 & m_s b\\ 0 & B & 0\\ m_s b & 0 & m_s m/M\end{bmatrix}; \quad G = \Omega\begin{bmatrix}0 & -\alpha & 0\\ \alpha & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \quad (22a,b)$$

$$K = \Omega^2\begin{bmatrix}C+m_s b^2-B & 0 & m_s b\\ 0 & C-A & 0\\ m_s b & 0 & k\end{bmatrix} \quad (22c)$$

where $\alpha=A+B-C$, $M=m+m_s$ (with $m_s \ll m$). The origin of the coordinate frame coincides with the CoM of the total system and the axes are the principal axes of the combined system with the spring–mass at the rest position ($\xi=0$) as shown in Fig. 4. We note that C is taken as the spin inertia of the rigid body.

The reduction of K gives

$$K_R = \Omega^2\begin{bmatrix}C+m_s b^2-B-\dfrac{m_s^2 b^2}{k}\Omega^2 & 0\\ 0 & C-A\end{bmatrix} \quad (23)$$

This expression is in agreement with the stability result given by Hughes [4] who established it by using the Routh–Hurwitz criterion for the roots. In fact, a spring–mass with damping d was added by Hughes [4] but this does not affect the stability limit.

The stability condition can be written as

$$C > B-m_s b^2(1-\Omega^2/\omega_s^2) \quad \text{with} \quad \omega_s^2 = k/m_s \quad (24)$$

where ω_s is the frequency of the mass–spring system.

In order to analyze this problem by means of the M&M model, we start with the total inertia matrix at equilibrium

$$J = \begin{bmatrix}J_{xx}=A+m_s(b^2+\xi^2) & 0 & 0\\ 0 & J_{yy}=B+m_s\xi^2 & J_{yz}=m_s b\xi\\ 0 & J_{yz} & J_{zz}=C+m_s b^2\end{bmatrix} \quad (25)$$

where ξ is the deformation variable. We can evaluate the elements J_{ij} with the help of the partials as in Eqs. (13) and (14)

$$\left.\frac{\partial^2 J_{zz}}{\partial\xi^2}\right|_{\xi=0} = 0 \quad (26a)$$

$$a_1 = \frac{\partial J_{xz}}{\partial \xi}\bigg|_{\xi=0} = 0; \quad b_1 = \frac{\partial J_{yz}}{\partial \xi}\bigg|_{\xi=0} = m_s b \tag{26b−c}$$

$$U = \frac{1}{2}k\xi^2 \Rightarrow \frac{\partial U}{\partial \xi^2}\bigg|_{\xi=0} = k \tag{26d}$$

Thus, we obtain for the elements c_{ij} of the correction matrix C_d

$$c_{11} = b^{\mathrm{T}}\Gamma^{-1}b; \quad c_{12} = c_{21} = c_{22} = 0 \tag{27a, b}$$

with

$$\Gamma = -\frac{1}{2}\frac{\partial^2 J_{zz}(q)}{\partial q^2}\bigg|_{q=0} + \frac{2}{\Omega^2}\frac{\partial^2 U(q)}{\partial q^2}\bigg|_{q=0} \tag{27d}$$

Finally, we find for the non-zero element c_{11} of C_d

$$c_{11} = m_s^2 b^2 \Omega^2/k \tag{28}$$

This produces the same stability condition as given in Eqs. (24) but without using the equations of motion. The full non-linear equations were investigated by Chinnery and Hall [21], in particular within the unstable domain. The complexity of the possible motions is surprising.

4.2. Rigid body appended with two cables

The second application considers the equations for a rigid body appended with two mass-less cables and two tip masses. This model has been used extensively in the 1970s for scientific satellites. The cables are attached at the height of the spacecraft CoM at a distance a from the nominal spin axis (z) and on the y-axis. Fig. 5 shows the yz-plane with the vertical deflection angles θ_i (i=1, 2). The tip mass is denoted by m_T. Each of the two cables has the fixed length l. The system contains also two equatorial deflection angles ψ_i in the xz-plane which are not considered in this example.

When splitting the equatorial and vertical deflection angles into their symmetric and anti-symmetric parts, we can see that the equatorial symmetric deflections couple to the attitude motion only when the attachment point is not at the height of the CoM. The equatorial anti-symmetric deflections modulate the spin rate, i.e., the so-called 'spin ripple' mode, but do not affect the spin-axis attitude. The vertical symmetric deflection angles couple only to the translations. Therefore, we consider here only the vertical anti-symmetric deflections which are denoted by the small angle θ_A, see Fig. 5. (If we include the other three deformation variables, we get into the situation described in the M&M section, i.e., Γ^{-1} is singular but the non-zero elements of a, b couple only to its non-singular elements.)

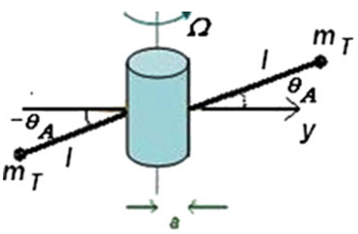

Fig. 5. Rigid body with two cables.

The corresponding 3×3 matrices of the system of equations for the variables $q^{\mathrm{T}}=(\theta_1, \theta_2, \theta_A)^{\mathrm{T}}$ are

$$M = \begin{bmatrix} A+A_T & 0 & i_c \\ 0 & B & 0 \\ i_c & 0 & 2m_T l^2 \end{bmatrix}; \quad G = \Omega\begin{bmatrix} 0 & -\alpha & 0 \\ \alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{29a, b}$$

$$k = \Omega^2\begin{bmatrix} C-B+C_r & 0 & i_c \\ 0 & C-A & 0 \\ i_c & 0 & i_c \end{bmatrix} \tag{29c}$$

with

$$a = A+B-C; \quad A_T = C_T = 2m_T(a+l)^2; \quad i_c = 2m_T l(a+l) \tag{30a−c}$$

We can now calculate the K_R matrix and the sufficient conditions for attitude stability as

$$K_R = \Omega^2\begin{bmatrix} C-B+C_T-i_c & 0 \\ 0 & C-A \end{bmatrix} \tag{31a}$$

$$C > A; \quad C > B-C_T+i_c = B-2m_T a(a+l) \tag{31b, c}$$

The result in Eq. (31c) shows that only a part, i.e., $2m_T a(a+l)$, of the inertia moment of the tip masses C_T is effective for the stability of the rigid-body inertia term C. Note also that C does not necessarily have to be the maximum-inertia axis. Concerning the system's moment of inertia C_s=C+C_T, we find that its stability is degraded because C_s must not only exceed B=B_s but also B+i_c. Finally, we mention that the term $2m_T a(a+l)\Omega^2\theta_A$ represents the torque acting on the rigid body for the deflection angle θ_A.

In order to apply the M&M method we introduce the small angle approximation for θ_A and find the results

$$J_{zz} = 2m_T(a+l-l\theta_A^2/2)^2 \approx 2m_T(a+l)^2 - 2m_T l(a+l)\theta_A^2 \tag{32a}$$

$$J_{xy} = J_{xz} = 0 \Rightarrow a_1 = 0 \tag{32b}$$

$$J_{yz} = -m_T l(a+l)\theta_A \Rightarrow b_1 = -2m_T l(a+l) \tag{32c}$$

$$c_{11} = -2m_T l(a+l); \quad c_{12} = c_{21} = c_{22} = 0 \tag{32d, e}$$

The result for c_{11} is consistent with the definition of i_c, see Eqs. (30c) and (31c).

Finally, we note that the two identical cables do not change the direction of the principal axes of the rigid body. However, if the cables are not at the same height as the CoM of the body, their vertical offset relative to the CoM of the system differs from the offset to the rigid-body CoM. The calculation of the new CoM, mass and inertia's is straightforward in this case.

In the failure case when only one cable has been deployed, the study of the equilibrium configuration in terms of the static deflection angles and new inertias is extremely complex. In case of a vertical offset of the attachment point of the cables, the symmetrical horizontal deflections couple to the attitude motion and the offset will be included in the stability condition.

5. Stability of MGKN systems

In practice, we must also assure the stability of a satellite while thrusting. The results of the previous sections, which are based on a torque-free satellite, are not applicable in this situation. A body-fixed thruster is a typical example of a follower force and the generic form of the linearized equations changes from MGK to MGKN where N is an anti-symmetric matrix. In practice this means that the constant matrix of the dynamic stiffness matrix is a general matrix which can always be split in symmetric and anti-symmetric parts. The most important physical difference is that MGKN systems are not conservative and can have a type of instability (flutter instability) that does not exist in conservative systems.

We present below two applications of a spinning satellite subject to a constant axial thrust. In the first example, the thrust is delivered by an apogee or perigee kick motor. In practice, these are solid rocket motors or liquid propellant engines. Such a motor or engine is, in principle at least, aligned with the nominal (i.e., geometric) spin axis. When all alignments are perfect, the thrust force will be exactly aligned with the actual (i.e., dynamic) spin axis and no torque acts on the satellite. In this ideal case, the CoM is accelerated and the uniform spin about the major or minor inertia axis remains the exact solution of the rotational motion. Therefore, the directional stability is the same as that of a free rigid body.

The nature of the mass and inertia variations during these burns are important. In the case of a liquid apogee engine, the propellant is guided to the satellite spin axis, which causes a spin-up of the satellite because the fuel's angular velocity decreases. In fact, the liquid propellant transfers its angular momentum to the body. In the case of a solid rocket motor, on the other hand, the gases swirl to the spin axis and maintain their angular momentum. Thus, there are no spin effects at all as predicted by van der Ha and Janssens [22] and as confirmed in practice.

In both applications we neglect mass variations even though these variations are important in the first example, i.e., the fourth-stage burn of the PAM-D/Ulysses combination. The second example deals with a North-South stationkeeping maneuver using an axial thruster in continuous mode on ESA's GEOS-I satellite in 1979. This model is also representative of an axial thruster that fails open.

5.1. Rigid body with axial thrust and particle on spin axis

We consider a particle on the spin axis of a symmetric satellite that is free to move in a plane perpendicular to the spin axis, see [23]. In this case, the axial acceleration, which is assumed to be constant, couples to the rotational motion. However, a uniform spin with the particle at rest still is an exact solution of the system. This example represents a simplification of the model used by Mingori and Yam [14,24] where the particle is connected to the spin axis by a radial spring to investigate the stability problem of the PAM-D series of apogee motors (under the accumulation of slag in an imbedded nozzle).

Within the E-frame, the thrust direction is variable since the thruster is body-fixed. The linearized equations of a model including such a follower force have the MGKN format where N represents the circulatory (anti-symmetric) matrix, see [11]. The derivation of these equations is very tedious and the final results (without spring) are as follows:

$$M=\begin{bmatrix} A' & 0 & 0 & -mh \\ 0 & A' & mh & 0 \\ 0 & mh & m(1-\mu) & 0 \\ -mh & 0 & 0 & m(1-\mu) \end{bmatrix} \tag{33a}$$

$$G=\Omega\begin{bmatrix} 0 & -(2A'-C) & -2mh & 0 \\ (2A'-C) & 0 & 0 & -2mh \\ 2mh & 0 & 0 & -2mh(1-\mu) \\ 0 & 2mh & 2mh(1-\mu) & 0 \end{bmatrix} \tag{33b}$$

$$K=\Omega^2\begin{bmatrix} C-A' & 0 & 0 & m(h+g/2\Omega^2) \\ 0 & C-A' & -m(h+g/2\Omega^2) & 0 \\ 0 & m(h+g/2\Omega^2) & -m(1-\mu) & 0 \\ -m(h+g/2\Omega^2) & 0 & 0 & -m(1-\mu) \end{bmatrix} \tag{33c}$$

$$N=\frac{mg}{2}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{33d}$$

The following input parameters are being used here: $q=(\theta_1, \theta_2, x_1, x_2)^{\mathrm{T}}$ is the state vector, where x_1 and x_2 are the particle's equatorial coordinates; $A'=A_b+mh^2$ is the transverse moment of inertia with respect to the CoM of the *system*; A_b is the transverse body inertia relative to the CoM of the body; M is the body mass, m is the particle mass, and $\mu=m/(m+M)$; h is the vertical coordinate of the particle with respect to the CoM of the system; $l=h/(1-\mu)$ is the coordinate of the particle relative to the CoM of the body; C is the spin inertia of the body and the system; Ω is the spin rate; g is the thrust acceleration, i.e., $F/(m+M)$.

Because the system is not conservative, the reduction method and the stability matrix cannot be used here. The only method available is to require that the roots of the characteristic equation are lying on the imaginary axis. This leads to the condition

$$mg(-l)\le C^2\Omega^2/(4A_b) \tag{34}$$

Eq. (34) provides a useful model for evaluating the instabilities experienced during the firings of the PAM-D boost motors in the 1980s. In particular, it was applied for checking the PAM-D stability during the injection of the ULYSSES satellite. In this application, the mass m refers to the maximum permissible slag mass that may accumulate within the PAM-D nozzle. Specific numerical values can be found in Ref. [23]. In reality, the actual slag mass during the ULYSSES injection is not known. No instability occurred during the injection but this may be due to the addition of an active control system on the PAM-D motor. Fig. 3 of Yam et al. [24] provides a stability curve $T_0(\beta)$, where T_0 is a non-dimensional quantity (defined in their Eq. (29)) and β is proportional to the spring constant k. The stability limit for a zero spring constant is $T_0(0)=-1/4$. We note that Eq. (34)

also shows that $l<0$ at the stability limit. This result implies that there is a limit to how far the particle may be located behind the CoM of the body.

When the intermediate variables occurring in T_0 are replaced by physical parameters (using their Eqs. (11), (15), (16), (22), and (24)) we find that $T_0(0)=-1/4$ is the same as in Eq. (34). When the criterion in Eq. (34) was derived by Janssens [23] it was thought that $k=0$ was the worst case for the stability limit.

However, Fig. 3 of Yam et al. [24] shows that, for small values of k, the distance that the particle may be behind the CoM shrinks and vanishes when k reaches the value such that

$$\omega_{res\text{-}k} = \omega_s (C/A_b)\sqrt{M/(M+m)} \tag{35a}$$

where

$$\omega_s = \sqrt{k/m} \tag{35b}$$

When $\omega_{res\text{-}k}$ reaches about 1.6 times the value of ω_s and the stability domain is the same as $k=0$.

The paper shows that the region where the undamped system is not unstable contains stable and unstable parts when dissipation is included. This is in line with the sufficient criteria given in [11] for the stability of MDGKN systems. All these highly mathematical criteria contain the D matrix. They are not well suited for practical applications and in practice one studies the roots of the characteristic equation.

5.2. Rigid body with axial thrusting and transverse torque

In the final example we consider a force vector that generates a transverse torque. The force may be produced by an apogee boost motor that is not perfectly aligned with the spin axis or by an axial thruster offset from the spin axis. In both cases, a constant non-zero perturbation torque $t=(t_x, t_y, t_z)^T$ is acting and the uniform pure-spin solution no longer satisfies the Euler equations

$$\begin{aligned} &A\dot{\omega}_1 + (C-B)\omega_2\omega_3 = t_x \\ &B\dot{\omega}_2 - (C-A)\omega_1\omega_3 = t_y \\ &C\dot{\omega}_3 + (B-A)\omega_1\omega_2 = t_z \end{aligned} \tag{36}$$

This problem is known as the 'self-excited' rigid body and has considerable practical relevance. In the case when all three torque components are present, there exists a stationary rotation if the torque components satisfy $t_x\, t_y\, t_z < 0$, see [25]. This stationary rotation has a unique fixed value for the angular velocity which is pointing perpendicular to both the torque direction and the angular momentum. Unfortunately, Eqs. (36) do not have a known analytical solution so we cannot linearize about it.

If only one non-zero torque component is present, a uniform spin solution exists in the plane perpendicular to the torque. When assuming a nominal spin about the z-axis (with small ω_1 and ω_2) the angular velocity makes an angle β relative to the z-axis in the y, z-plane when a torque t_x is applied (and an angle α in the xz-plane for a torque t_y) with the following relationships:

$$\tan\beta = t_x/\{\Omega^2(C-B)\}; \quad \tan\alpha = -t_y\{\Omega^2(C-A)\} \tag{37a, b}$$

The larger the nominal spin Ω is, the smaller the deviation angles α and β. When there is a single cross-inertia i_{yz} or i_{xz} in the reference system, a uniform spin is possible about the new principal axis direction

$$[\sin\alpha, 0, \cos\alpha]^T \quad with \quad \tan\alpha = i_{xz}/(C-A) \tag{38a}$$

$$[0, \sin\beta, \cos\beta]^T \quad \text{with} \quad \tan\beta = i_{yz}/(C-B) \tag{38b}$$

In the small-angle approximation, the angles α and β play the same role in different planes. Therefore, if only one transverse torque component is applied, its effect is the same as that of a cross-inertia i_{yz} or i_{xz}

When both transverse torque components are present and $t_z=0$, a constant bounded spin rate is no longer an exact solution of Eqs. (36) for an asymmetric body. Therefore, we can only do a linearization about the initial state vector $(0, 0, \Omega)^T$ with $\Omega=\Omega_3(0)$

$$\begin{aligned} &A\dot{\omega}_1 + (C-B)\Omega\omega_2 = t_x \\ &B\dot{\omega}_2 - (C-A)\Omega\omega_1 = t_y \\ &C\dot{\omega}_3 = 0 \end{aligned} \tag{39a–c}$$

We find in first-order approximation that $\Omega_3(t) \approx \Omega$ is constant.

When introducing the Tait–Bryan angles via the matrix in Eq. (1), we obtain an MGKN system with a constant right-hand side $t=(t_x, t_y, 0)^T$

$$M\ddot{q} + G\dot{q} + (K+N)q = t \tag{40a}$$

with the anti-symmetric matrix N defined by

$$N = \begin{bmatrix} 0 & 0 & t_y \\ 0 & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix} \tag{40b}$$

We discuss first the system in Eqs. (39). The solution of this non-homogeneous system is easily found as

$$\omega_1 = t_x \sin(\nu\Omega t)/(\nu A\Omega) - t_y\{1-\cos(\nu A\Omega)\}/[(C-A)\Omega] \tag{41a}$$

$$\omega_2 = t_x\{1-\cos(\nu\Omega t)\}/[(C-B)\Omega] + t_y \sin(\nu\Omega t)/(\nu B\Omega) \tag{41b}$$

with

$$\nu^2 = (C-B)(C-A)/AB > 0 \quad \text{when} \quad C > \{A,B\} \text{ or } C < \{A,B\} \tag{41c}$$

The transverse angular velocity describes an ellipse with center at $\{\alpha\Omega, \beta\Omega\}$, i.e., the constant terms of Eqs. (41a,b), which are the same as in Eq. (37) for small α and β. The average values of the rates ω_1 and ω_2 over a nutation period also give precisely the same results.

However, when inserting these averaged values into the third complete Euler equation we obtain a 'secular' change in the spin rate

$$C\dot{\Omega}_3 \approx \frac{t_x t_y}{\Omega}\frac{(B-A)}{(C-A)(C-B)} \tag{42}$$

The corresponding mean rate of change of energy is given by

$$\dot{E} \approx \frac{t_x t_y}{\Omega}\frac{(B-A)}{(C-A)(C-B)} \tag{43}$$

and is different from zero when the body is not symmetric and both transverse torque components are present. When both torque components have the same sign, the satellite spins up. When they have opposite signs, the satellite will be despun completely after a time (see [15])

$$\tau \approx C\Omega_0^2/(3\overline{\dot{E}}) \tag{44}$$

This instability result is independent of the magnitude of t_x and t_y and therefore contradicts the application of the ERB method to the case of a misaligned apogee motor. The GEOS-I satellite narrowly escaped this fate.

More specific details and numerical parameters on the GEOS-1 despin instability can be found in Table 3 on p. 274 of Ref. [15]. The fact that GEOS-I had two long cables appended was not relevant for this type of instability. The analysis can easily be adapted to include the cables. The paper by van der Ha [16] presents a second-order perturbation expansion solution of Eqs. (39), which confirmed this instability. This is a perfect example to illustrate that a bounded solution based on a linearization about an initial state does not guarantee the stability of the full non-linear system.

Because the right-hand side of (40a) is constant, we can introduce a potential $U_T = \theta_1 t_x + \theta_2 t_y$ so that the equations can be derived from a Lagrangian or Hamiltonian formalism. However, the conserved quantity is no longer a quadratic form as the new term is linear. Hence, the usual stability theorems, which are all based on quadratic Lyapunov functions, are not applicable.

6. Conclusions

The use of stability theories has always been an important issue in the design and operations of satellites. The most widely used tool was to study the roots of a linearized set of dynamic equations that did not include damping. These equations may result from a linearization about a reference motion or from an initial state. Even in the first case, we cannot conclude about the stability of the non-linear system but damping will be beneficial. In the second case the first-order results may be invalid as illustrated by GEOS-I in-orbit behavior.

For spinning satellites, the notion of directional stability, with respect to the two attitude variables only, emerges as the relevant model for practical applications. This kind of stability can now be investigated directly by means of an appropriate inertia tensor (in the force-free case). When the linearized equations of motion are available, the stability matrix can immediately be obtained from the Frobenius–Schur formula for the reduction of partitioned matrices. Under axial thrusting, the reference uniform spin motion may or may not exist. In the first case, Lypunov's indirect method is applicable. In the second case, the directional stability cannot be concluded from the linearized equations about an initial state. We have presented a few examples of practical relevance that demonstrate the application of the various methods.

Appendix A. Introduction to MGK system

We consider a Lagrangian system with multi-dimensional generalized coordinates q and linear velocity terms

$$L = \frac{1}{2}\dot{q}^{\mathrm{T}}M\dot{q} + \frac{1}{2}\dot{q}^{\mathrm{T}}Gq - \frac{1}{2}q^{\mathrm{T}}Kq \tag{A1}$$

The resulting equations of motion follow from the Lagrangian in Eq. (A1) after calculating the momenta p

$$\left.\begin{aligned} p = \frac{\partial L}{\partial \dot{q}} &= M\dot{q} + \frac{1}{2}Gq \\ \frac{\partial L}{\partial q} &= -\frac{1}{2}G\dot{q} - Kq \end{aligned}\right\} \Rightarrow M\ddot{q} + G\dot{q} + Kq = 0 \tag{A2a−c}$$

The gyroscopic matrix G characterizes spinning systems. Because the system is homogeneous, it has the null solution $q=0$. The stability of the null solution can be investigated by using the characteristic equation of a polynomial eigenvalue problem or by Lyapunov's direct method as for linear systems there is a systematic way of constructing Lyapunov functions.

The energy of the system in terms of the coordinates q and the velocities $\dot{q}$ is given by

$$E = \frac{1}{2}\dot{q}^{\mathrm{T}}M\dot{q} + \frac{1}{2}q^{\mathrm{T}}Kq \tag{A3}$$

The matrix G does not appear in the energy equation because the gyroscopic force $G\dot{q}$ is perpendicular to the velocity $\dot{q}$ and does not perform any work on the system. A direct substitution of the momenta p from Eq. (A2a) into the energy Eq. (A3) yields the Hamiltonian H

$$E = H = \frac{1}{2}p^{\mathrm{T}}M^{-1}p - \frac{1}{2}p^{\mathrm{T}}M^{-1}Gq + \frac{1}{2}g^{\mathrm{T}}\left[K - \frac{1}{4}GM^{-1}G\right]q \tag{A4}$$

The energy in Eq. (A3) is undoubtedly a Lyapunov function when $K>0$ so that the system is Lyapunov stable in this case. When $K<0$, the quadratic form in the second term of Eq. (A3) will be negative and the energy may or may not be a positive definite function. Therefore, the possibility exists that the system is Lyapunov stable for $K<0$.

A straightforward instability criterion due to Hagedorn [13], is available

$$\left[K - \frac{1}{4}GM^{-1}G\right] < 0 \tag{A5}$$

The calculation of $\dot{E}$ (using $\dot{q}^{\mathrm{T}}Gq = 0$) along a trajectory gives, as expected, $\dot{E} = 0$ and not $\dot{E} < 0$, so that the system is not asymptotically stable.

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