Analytical Models for Relative Motion Under Constant Thrust

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A general formulation for the relative motion allowing for arbitrary perturbing or thrust forces on each of the two satellites is presented. Exact as well as approximate perturbation solutions for the linearized relative motion equations with constant radial or circumferential forces acting on the subsatellite are established. The validity and usefulness of these solutions is assessed for a few realistic applications related to the European Space Agency's retrievable carrier (EURECA) rendezvous maneuvering with the Shuttle. The results are of general interest for the fast calculation of relative subsatellite motion under thrust forces.

I. Introduction

THE study of orbital relative motion and rendezvous problems is becoming increasingly important, mainly because of space station applications. These types of problems have been studied since about 1960, and relatively simple analytical expressions for relative motion and rendezvous using impulsive maneuvers have been available since that time. In actual practice, however, maneuvers during rendezvous operations cannot normally be considered impulsive (because of the low thrust levels used); therefore, finite-thrust arcs must be studied. Of course, impulsive maneuvers often serve a very useful role as starting or reference solutions for more accurate approaches.

In the present paper, new analytical models for relative motion under constant circumferential and radial thrust (on the probe vehicle) are formulated. The rationale of the present theory is to obtain a better understanding of the nature and the possibilities of relative motion during constant thrust arcs. This knowledge would be useful for designing and evaluating orbit rendezvous strategies.

Many spacecraft are equipped with Earth sensors, which allow the spacecraft to keep a fixed orientation relative to the Earth by means of onboard attitude control. The Earth-pointing attitude orientation leads naturally to the circumferential (relative to the spacecraft orbit) thrust direction. This is the case, for instance, for the European Space Agency's retrievable carrier (EURECA), when it is performing rendezvous maneuvers with the Shuttle during its retrieval phase. An optimal rendezvous strategy under the constraint of a circumferential thrust direction would consist of a succession of three types of arcs: positive- and negative-thrust arcs, and coast arcs. The resulting relative motion during each of these arcs can be described by the analytical models presented in the present paper.

Exact solutions will be formulated for the linearized (in terms of ratio of relative to orbital distances) equations during circumferential and radial-thrust arcs. In addition, specific approximate asymptotic solutions are presented for these linearized equations in the case when the thrust/gravity ratio is sufficiently small so that the conditions of perturbation theory apply. The solutions to be presented are valid for close subsatellite motion relative to a circular and coplanar station orbit.

The solutions obtained would be useful for analysis and design of relative motion and rendezvous trajectories. Because of their relative compactness, the analytical results established here could also be suitable for incorporation in autonomous terminal maneuver planning software.

II. Equations of Relative Motion

A. General Formulation

The equations of relative motion follow from Newton's second law by subtracting the individual equations of motion for the two satellites in an inertial reference frame:

$$\ddot{\boldsymbol{\rho}} + \mu\left[(\boldsymbol{r}+\boldsymbol{\rho})/|\boldsymbol{r}+\boldsymbol{\rho}|^3 - \boldsymbol{r}/r^3\right] = \boldsymbol{F}_p/m_p - \boldsymbol{F}/m \tag{1}$$

The relative position vector $\boldsymbol{\rho}$ is defined as $\boldsymbol{r}_p - \boldsymbol{r}$ as shown in Fig. 1.

Parameters belonging to the probe (i.e., subsatellite) have a subscript p, whereas those associated with the station have no subscript. The ideal central-body gravity forces are incorporated on the left-hand side of Eq. (1) with μ denoting the Earth's gravitational parameter. All other forces (induced by perturbations or thrust) acting on each of the two satellites are contained in $\boldsymbol{F}$ and $\boldsymbol{F}_p$, respectively.

The equations of motion can be expanded in components in the local x, y, z reference frame, i.e., the osculating frame at-

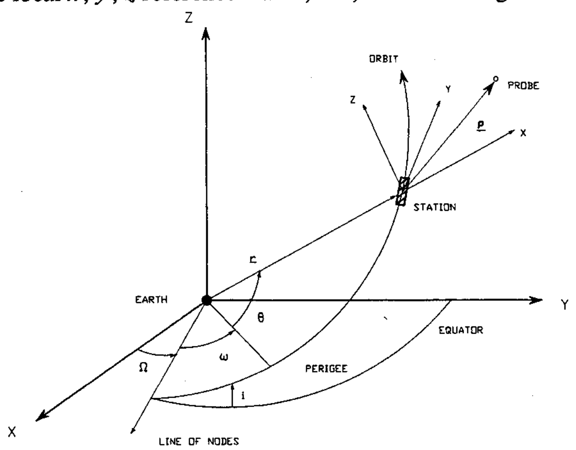

Fig. 1 Visualization of orbital geometry.

Received July 10, 1988; presented as Paper 88-4300 at the AIAA/AAS Astrodynamics Conference, Minneapolis, MN, Aug. 15-17, 1988; revision received Jan. 6, 1989.

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tached to the station's orbit, by using Eq. (12) of Ref. 1:

$$\ddot{\rho} = [\ddot{\rho}]_{\rm loc} + 2W \times [\dot{\rho}]_{\rm loc} + W \times (W \times \rho) + \dot{W} \times \rho \qquad (2)$$

The terms appearing here are known as relative, Coriolis, centripetal, and transverse accelerations, respectively. The vector W designates the instantaneous rate of rotation of the local frame relative to inertial space. By the nature of the definition of the osculating plane, the y-component of the rotation vector W must vanish (see for instance Refs. 2 and 3). Therefore, W can be written as

$$W = wx + \dot{\nu}z; \qquad w = rF_z/(mh); \qquad \dot{\nu} = h/r^2 \qquad (3)$$

Here, h stands for the orbital angular momentum (per unit mass). Throughout the paper, the notation x, y, z will be used for the unit vectors along the local x, y, z axes. Finally, the following relative equations of motion are obtained from Eqs. (1-3)

$$\ddot{x} - 2\dot{y}\dot{\nu} - x\dot{\nu}^2 - y\ddot{\nu} + \mu(r + x)/r_p^3 - \mu/r^2 = f_x \qquad (4a)$$

$$\ddot{y} + 2\dot{x}\dot{\nu} - y\dot{\nu}^2 + x\ddot{\nu} + \mu y/r_p^3 = f_y \qquad (4b)$$

$$\ddot{z} + \mu z/r_p^3 = f_z \qquad (4c)$$

The accelerations appearing on the right-hand sides of Eqs. (4) actually take account of all external forces acting on both satellites

$$f_x = F_{px}/m_p - F_x/m - z\dot{\nu}w \qquad (5a)$$

$$f_y = F_{py}/m_p - F_y/m + 2\dot{z}w + yw^2 + z\dot{w} \qquad (5b)$$

$$f_z = F_{pz}/m_p - F_z/m - 2\dot{y}w + w(zw - x\dot{\nu}) - y\dot{w} \qquad (5c)$$

It should be noted that F_{px}, F_{py}, F_{pz} designate the components of the probe's perturbing forces expanded along the local axes belonging to the station. Since these forces will usually be known relative to a coordinate frame attached to the probe itself, a transformation using an inertial reference frame as an intermediary is required.

When denoting the attitude matrices of the station's and probe's local reference frames (relative to an inertial frame) by $[A]$ and $[A_p]$, respectively, it can be shown that

$$F_p = [A][A_p]^T F_{p,\rm loc} \qquad (6)$$

B. Angular Formulation

The equations of relative motion will now be formulated in terms of the quasi-angle ν, defined by its differential equation $\dot{\nu} = h/r^2$ as an independent variable. In this manner, one arrives at an attractive formulation where the differential equations of motion are expressed in terms of the new independent variable ν. In the absence of perturbations or thrust forces, the selected independent variable coincides with the well-known true anomaly. In the presence of such forces, the definition of ν as given in Eqs. (3) is still meaningful (see for instance Ref. 3).

Further significant simplification of the equations of motion can be achieved by the normalization of the relative position coordinates using the station's orbital radius r: $\xi = x/r$, $\eta = y/r$, $\zeta = z/r$.

First, the polar form of the station's equations of motion is recalled:

$$\ddot{r} - r\dot{\nu}^2 + \mu/r^2 = F_x/m, \qquad \dot{h} = rF_y/m \qquad (7)$$

These equations are representative of the station's orbital motion under arbitrary perturbing or thrust forces. They form the starting point for the derivation of more convenient representations in terms of orbital elements as shown in Refs. 2 and 3. The station's motion can in principle be obtained from Eqs. (7) in isolation of the probe's relative motion to be studied below. Of course, the characteristics of the station's orbital motion will appear in the results of the relative motion.

The transformation from t to ν as independent variable uses the following relationships:

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\dot{\nu}\mathrm{d}}{\mathrm{d}\nu}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}t^2} = \ddot{\nu}\frac{\mathrm{d}}{\mathrm{d}\nu} + \frac{\dot{\nu}^2\mathrm{d}^2}{\mathrm{d}\nu^2} \qquad (8)$$

with

$$\dot{\nu} = h/r^2, \qquad \ddot{\nu} = h^2(h'/h - 2r'/r)/r^4 \qquad (9)$$

where $'$ stands for $\mathrm{d}/\mathrm{d}\nu$. After a considerable amount of algebra involving Eqs. (4) and (7-9), one can establish the following result for the equations of relative motion:

$$\xi'' - 2\eta' + (1 + \xi)R(\nu) = a_\xi \qquad (10a)$$

$$\eta'' + 2\xi' + \eta R(\nu) = a_\eta \qquad (10b)$$

$$\zeta'' + \zeta + \zeta R(\nu) = a_\zeta \qquad (10c)$$

with R defined as

$$R(\nu) = \mu r(r^3/r_p^3 - 1)/h^2 \qquad (11)$$

The forcing terms are defined as follows:

$$a_\xi = r^3\Big[f_x - \xi F_x/m - (\xi' - \eta)F_y/m\Big]\Big/h^2 \qquad (12a)$$

$$a_\eta = r^3\Big[f_y - \eta F_y/m - (\eta' + \xi)F_y/m\Big]\Big/h^2 \qquad (12b)$$

$$a_\zeta = r^3\Big[f_z - \zeta F_z/m - \zeta' F_y/m\Big]\Big/h^2 \qquad (12c)$$

The former two of the equations appearing in Eqs. (10) have been derived previously in Eqs. (20) of Ref. 1 by a different approach. It may be emphasized that the system of Eqs. (10) are completely equivalent to the original system of Eqs. (1) as no approximations whatsoever have been introduced up to this stage.

C. Linearized Equations

In the case where the relative distance is small compared to the orbital radii, it may be justified to neglect terms of second-order smallness in terms of the ratio relative/orbital distance (i.e., ρ/r). The function $R(\nu)$ of Eq. (11) can be expanded as follows:

$$h^2R(\nu)/(\mu r) = \Big[(1 + \xi)^2 + \eta^2 + \zeta^2\Big]^{-1.5} - 1$$

$$\cong -3\xi + 6\xi^2 - 3(\eta^2 + \zeta^2)/2 + \mathcal{O}(\rho^3/r^3) \qquad (13)$$

This expansion results in the following system of equations:

$$\xi'' - 2\eta' - 3(\mu r/h^2)\xi = a_\xi \qquad (14a)$$

$$\eta'' + 2\xi' = a_\eta \qquad (14b)$$

$$\zeta'' + \zeta = a_\zeta \qquad (14c)$$

This is a linear system with (in general) periodic coefficients, and it has been studied extensively by Tschauner and Hempel.[4,5]

A further simplification is introduced by considering an ideal circular station orbit that can only be approximately true in actual practice because of perturbing influences. When it is assumed that r remains constant (equal to $r(\nu) = h^2/\mu$), the first of Eqs. (14) can be further simplified to become

$$\xi'' - 2\eta' - 3\xi = a_\xi \tag{15}$$

This system corresponds to the well known Clohessy-Wiltshire equations[6] and represents a convenient starting point for relative motion and rendezvous analyses.

In line with the assumption that r is constant, the force components F_x and F_y that are acting on the station must vanish. This implies that the forcing terms can be simplified accordingly [using Eqs. (12) and (5)]

$$a_\xi = r^3 f_x/h^2, \qquad a_\eta = r^3 f_y/h^2 \tag{16a}$$

$$a_\zeta = r^3(f_z - \zeta F_z/m)/h^2 \tag{16b}$$

It may be noted that the out-of-plane force component F_z, which induces a slow rotation of the station's orbit plane, can still be included in this model as it does not affect the in-plane motion [see Eqs. (3) and (7)]. If F_z also vanishes, the rotation w of the orbit plane disappears so that only the forces acting on the probe (i.e., F_p) will be remaining in the model, as can be seen from Eqs. (5).

III. Thrust Forces

Under the conditions stated at the end of the previous section, only the forces acting on the probe enter the analysis. These forces will be assumed constant over the interval under consideration. This is a realistic assumption for typical thrust forces acting on the probe vehicle. Equation (6) provides the transformation of forces from the probe's local reference frame to the components of F_p expressed in the station's local frame. In general, this transformation would involve rather complicated functions of the differences in angular elements such as $\Delta\nu$, Δi, $\Delta\Omega$ between the two orbits. If it is now assumed that the two orbit planes essentially coincide (e.g., after completion of an out-of-plane maneuver), this transformation can be reduced to a simple rotation over $\Delta\nu$.

A. Circumferential Thrust

First, it is assumed that a constant circumferential thrust T_c is acting on the probe vehicle. In the case where the aforementioned conditions apply, the thrust components in the station's local frame can be expressed as (see also Fig. 2)

$$\boldsymbol{F}_p = T_c[\cos\Delta\nu\, \boldsymbol{y} - \sin\Delta\nu\, \boldsymbol{x}] \tag{17}$$

Expressing this in terms of the normalized (relative to r) relative coordinates ξ, η, and ζ, one finds

$$\sin\Delta\nu = y/r_p \cong \eta\left[1 - \xi + O(\xi^2, \eta^2)\right] \tag{18a}$$

$$\cos\Delta\nu = (r + x)/r_p \cong (1+\xi)\left[1 - \xi + O(\xi^2, \eta^2)\right] \tag{18b}$$

Here, $O(\ldots)$ refers to omitted second- and higher-order terms in the normalized relative distance. In a linear theory, one can take the approximations $\sin\Delta\nu \cong \eta$, and $\cos\Delta\nu \cong 1$ with errors in the order of the square of the normalized relative distance. It will be assumed that the thrust force is the only force acting on the probe. Since the free motion in the z-direction is well known, only the in-plane motion will be considered here.

Under the assumptions expressed previously, the Clohessy-Wiltshire equations can be extended to incorporate the circumferential thrust acceleration on the probe [see Eqs. (14) and (15)]:

$$\xi'' - 2\eta' - 3\xi = -\epsilon\eta \tag{19a}$$

$$\eta'' + 2\xi' = \epsilon \tag{19b}$$

with $\epsilon = r^2 T_c/(\mu m)$. It should be emphasized that no expansion in terms of ϵ has taken place in Eqs. (19). Since all variables appearing in the system of Eqs. (19) have been properly nondimensionalized, the parameter ϵ is representative of the effectiveness of the thrust relative to that of the gravity force.

As an example, the acceleration delivered by 70-N thrusters on the EURECA spacecraft is 0.0206 m/s^2, on the basis of a mass of 3400 kg. This results in a value of $\epsilon = 0.0024$. By taking ϵ positive, negative, or zero, one can make the system of Eqs. (19) valid for any of the two possible circumferential thrust arcs as well as for a coast arc.

B. Radial Thrust

A system of equations similar to Eqs. (19) can be derived in the case that a constant radial thrust T_r is acting on the probe. The radial thrust components in the station's local frame may be expressed as

$$\boldsymbol{F}_p = T_r\,(\sin\Delta\nu\, \boldsymbol{y} + \cos\Delta\nu\, \boldsymbol{x}) \tag{20}$$

Under the same assumptions as in the previous section, it can be shown that the Clohessy-Wiltshire equations incorporating radial thrust acceleration on the probe are of the form

$$\xi'' - 2\eta' - 3\xi = \epsilon \tag{21a}$$

$$\eta'' + 2\xi' = \epsilon\eta \tag{21b}$$

with $\epsilon = r^2 T_r/(\mu m)$. By taking ϵ positive, negative, or zero, one can make this system valid for any of the two possible radial thrust arcs as well as for a coast arc.

IV. Exact Analytical Solutions

A. Circumferential Thrust

The exact analytical solution for the system of Eqs. (19) can be obtained in a direct manner. First, a new variable $\gamma(\nu)$ is introduced as

$$\gamma(\nu) = \eta'(\nu) + 2\xi(\nu) \tag{22}$$

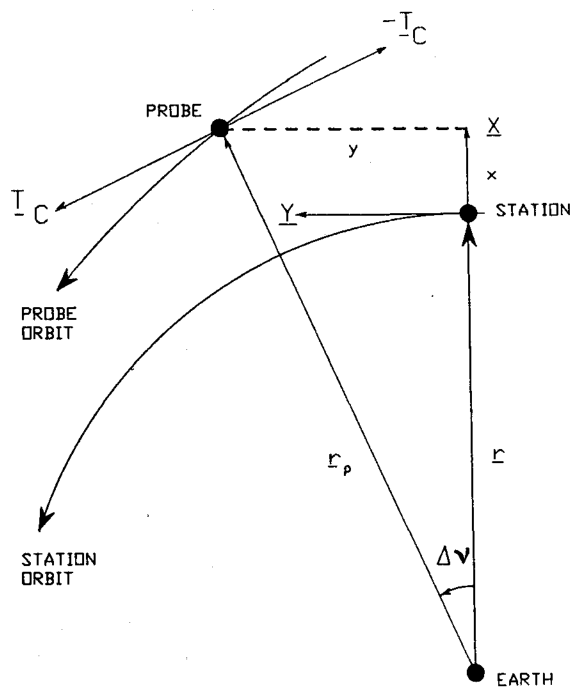

Fig. 2 Visualization of circumferential thrust.

which leads to $\gamma'(\nu)=\epsilon$ according to the latter equation of the system [Eqs. (19)]. Therefore, one finds immediately

$$\gamma(\nu)=\gamma_0+\epsilon\nu, \qquad (\gamma_0=\eta_0'+2\xi_0) \tag{23}$$

Differentiating the former equation in Eqs. (19), and substituting the latter, one finds

$$\xi^{(3)}+\xi'+\epsilon\eta'=2\epsilon \tag{24}$$

where the term η' can be readily eliminated with the aid of Eqs. (22) and (23); therefore Eq. (24) becomes

$$\xi^{(3)}+\xi'-2\epsilon\xi=\epsilon(2-\gamma_0)-\epsilon^2\nu \tag{25}$$

This is a decoupled linear equation with the independent variable appearing in the right-hand side. A particular solution ξ_p that satisfies Eq. (25) exactly can be obtained by inspection:

$$\xi_p(\nu)=(2\gamma_0-3)/4+\epsilon\nu/2 \tag{26}$$

The general solution of the homogeneous part is calculated from the corresponding characteristic polynomial equation

$$\lambda_i^3+\lambda_i-2\epsilon=0 \qquad (i=1,2,3) \tag{27}$$

This cubic equation can be solved by means of a standard technique

$$\lambda_1=-2\sqrt{1/3}\cot(2\alpha), \quad \lambda_{2,3}=\sqrt{1/3}\cot(2\alpha)\pm i\csc(2\alpha) \tag{28}$$

with the auxiliary angle α is defined as

$$\alpha=\arctan\left\{\mp|\tan(\beta/2)|^{1/3}\right\} \tag{29a}$$

$$\beta=\arctan\left\{-(1/3)^{3/2}/\epsilon\right\} \tag{29b}$$

The sign of α is defined by the sign of ϵ: the upper sign holds for $\epsilon>0$ and the lower sign for $\epsilon<0$. It is seen that the conditions $|\beta|\le\pi/2$ and $|\alpha|\le\pi/4$ are satisfied so that the resulting roots in Eqs. (28) are meaningful.

The homogeneous solution is exact as no approximation for small ϵ has been applied anywhere. Naturally, it should not be overlooked that the system of Eqs. (19) is an approximation in the sense that the relative distance is assumed to be small.

The exact solution for $\eta(\nu)$ can be obtained from $\xi(\nu)$ by integration using Eqs. (22) and (23). The complete exact solutions of Eqs. (19) are now obtained in the form:

$$\xi(\nu)=\xi_p(\nu)+A_1\exp(\lambda_1\nu) + \exp(-\lambda_1\nu/2)\Big[B_1\cos(\omega\nu)+B_2\sin(\omega\nu)\Big] \tag{30a}$$

$$\begin{aligned}\eta(\nu)&=\eta_0+3\nu/2-2A_1\Big[\exp(\lambda_1\nu)-1\Big]\Big/\lambda_1+\\ &-\Big[B_1\lambda_1+B_2\omega+(2B_1\omega-B_2\lambda_1)\sin(\omega\nu)+\\ &-(B_1\lambda_1+2B_2\omega)\cos(\omega\nu)\exp(-\lambda_1\nu/2)\Big]\Big/(\omega^2+\lambda_1^2/4)\end{aligned} \tag{30b}$$

The frequency ω is defined as $\csc(2\alpha)$, whereas the integration constants A_1, B_1, B_2 are fairly complicated expressions in terms of the initial conditions and their derivatives:

$$B_1=\left\{-\xi_0''-\lambda_1\xi_0'+\epsilon\lambda_1/2+\lambda_1^2(3/2-\eta_0')\right\}\Big/(\omega^2+9\lambda_1^2/4) \tag{31a}$$

$$A_1=3/4-\eta_0'/2-B_1 \tag{31b}$$

$$B_2=\left\{\xi_0'-\epsilon/2+\lambda_1(3B_1-3/2+\eta_0')/2\right\}\Big/\omega \tag{31c}$$

Note that $\xi_0''=3\xi_0+2\eta_0'-\epsilon\eta_0$ as can be seen from Eqs. (19).

B. Radial Thrust

The exact analytical solutions of the system of Eqs. (21) is obtained as follows. Differentiating the former equation in Eqs. (21), and substituting the latter, one finds

$$\xi^{(3)}+\xi'-2\epsilon\eta=0 \tag{32}$$

Differentiating again and using the first equation of the system in Eqs. (21)

$$\xi^{(4)}+(1-\epsilon)\xi''+3\epsilon\xi=-\epsilon^2 \tag{33}$$

A particular solution of Eq. (33) is simply $\xi_p=-\epsilon/3$. The general solution of the homogeneous part is calculated from the corresponding characteristic polynomial equation of fourth degree

$$\lambda_i^4+(1-\epsilon)\lambda_i^2+3\epsilon=0, \qquad (i=1,\ldots,4) \tag{34}$$

with the four roots

$$\lambda_{1,2}=\pm i\Big[(1-\epsilon)+\sqrt{(1-\epsilon)^2-12\epsilon}\Big]^{1/2}\Big/\sqrt{2} \tag{35a}$$

$$\lambda_{3,4}=\pm i\Big[(1-\epsilon)-\sqrt{(1-\epsilon)^2-12\epsilon}\Big]^{1/2}\Big/\sqrt{2} \tag{35b}$$

For small values of ϵ, it can be shown that $\lambda_{1,2}$ generates periodic solutions with frequency close to the orbital frequency, whereas $\lambda_{3,4}$ leads to long-term periodic solutions with periods of about $2\pi/(3\epsilon)^{1/2}$.

The solution for $\eta(\nu)$ follows from repeated differentiation of $\xi(\nu)$ as can be seen from Eq. (32). The complete exact solution can finally be written in the form

$$\xi(\nu)=-\epsilon/3+C_1\cos(\omega_1\nu)+S_1\sin(\omega_1\nu) + C_2\cos(\omega_2\nu)+S_2\sin(\omega_2\nu) \tag{36a}$$

$$\begin{aligned}\eta(\nu)=\,&\omega_1(1-\omega_1^2)\Big\{S_1\cos(\omega_1\nu)-C_1\sin(\omega_1\nu)\Big\}\\ &+\omega_2(1-\omega_2^2)\Big\{S_2\cos(\omega_2\nu)-C_2\sin(\omega_2\nu)\Big\}\Big/(2\epsilon)\end{aligned} \tag{36b}$$

where the constants ω_1 and ω_2 are defined as

$$\omega_1=\Big[(1-\epsilon)+\sqrt{(1-\epsilon)^2-12\epsilon}\Big]^{1/2}\Big/\sqrt{2} \tag{37a}$$

$$\omega_2=\Big[(1-\epsilon)-\sqrt{(1-\epsilon)^2-12\epsilon}\Big]^{1/2}\Big/\sqrt{2} \tag{37b}$$

The integration constants C_1, S_1, C_2, S_2 can be expressed in terms of the initial conditions and their first-order derivatives:

$$C_2=\Big\{(\omega_1^2+3)(\xi_0+\epsilon/3)+2\eta_0'\Big\}\Big/(\omega_1^2-\omega_2^2) \tag{38a}$$

$$S_2=\Big\{(\omega_1^2-1)\xi_0'+2\epsilon\eta_0\Big\}\Big/\Big[\omega_2(\omega_1^2-\omega_2^2)\Big] \tag{38b}$$

$$C_1=\xi_0+\epsilon/3-C_2 \tag{38c}$$

$$S_1=\xi_0'/\omega_1-\omega_2S_2/\omega_1 \tag{38d}$$

V. Approximate Analytical Solutions

A. Circumferential Thrust

The exact analytical solution derived in Eqs. (30) is somewhat complex for routine usage, especially in the calculation of ω and λ_1. Therefore, it may be of interest to investigate the possibility of simplifying the results, for instance by means of asymptotic expansions in the small parameter ϵ.

First, Eqs. (29) are analyzed as to their asymptotic properties. By writing $\delta=\beta\pm\pi/2$ (with upper and lower signs for $\epsilon>$ and <0, respectively), it can be assured that δ is small, and so from Eqs. (29) one gets

$$\delta=\arctan\left\{1/(-\tan\beta)\right\}$$
$$=\arctan(3\epsilon\sqrt{3})\cong 3\epsilon\sqrt{3}+\mathcal{O}(\epsilon^2) \quad (39)$$

Subsequently, α is expressed in terms of δ using

$$\tan(\beta/2)=\tan(\delta/2\mp\pi/4)$$
$$\cong \mp 1+\delta\mp\delta^2/2+\mathcal{O}(\delta^3) \quad (40)$$

so that

$$\mp|\tan(\beta/2)|^{1/3}\cong\mp 1+\delta/3\mp\delta^2/18+\mathcal{O}(\delta^3) \quad (41)$$

This result implies that α is close to $\mp\pi/4$, and so α can be expressed as $\alpha=\Delta\mp\pi/4$ with small Δ. Therefore, one can expand:

$$\tan\alpha=\left[\sin(2\Delta)\mp 1\right]/\cos(2\Delta)$$
$$\cong\mp 1+2\Delta\mp 2\Delta^2+\mathcal{O}(\Delta^3) \quad (42)$$

Since Eqs. (41) and (42) must be identical according to Eqs. (29), it follows that $\Delta\equiv\delta/6$ and, finally,

$$\alpha=\delta/6\mp\pi/4\cong\epsilon\sqrt{3}/2\mp\pi/4+\mathcal{O}(\epsilon^3) \quad (43)$$

This equation is very useful as it provides a direct (but approximate) connection between ϵ and the corresponding α. It may be noted that the result is valid for ϵ negative or positive. The exponents in Eqs. (30) can now be approximated as

$$\lambda_1\cong 2\sqrt{1/3}\,\tan\left[\epsilon\sqrt{3}+\mathcal{O}(\epsilon^3)\right]\cong 2\epsilon+\mathcal{O}(\epsilon^3) \quad (44a)$$

$$\omega=\csc(2\alpha)\cong\mp(1+3\epsilon^2/2)+\mathcal{O}(\epsilon^3) \quad (44b)$$

With the aid of these asymptotic results it is possible to expand the integration constants A_1, B_1, B_2 of Eqs. (31) in terms of powers of ϵ up to second order.

After substituing the expansions of these constants in the relevant Eqs. (30), the approximate solutions up to the first order (in terms of ϵ) become

$$\xi^{(1)}(\nu)\cong 2\xi_0'-\eta_0+(6\xi_0+3\eta_0'+2)\nu$$
$$+\left[\eta_0-2\xi_0'+(3\xi_0+2\eta_0')\nu\right]\cos\nu+$$
$$-\left[9\xi_0+5\eta_0'+2+\xi_0'\nu\right]\sin\nu \quad (45a)$$

$$\eta^{(1)}(\nu)\cong 2(12\xi_0+7\eta_0'+2)+2(\eta_0-2\xi_0')\nu+$$
$$-(6\xi_0+3\eta_0'+3/2)\nu^2+$$
$$-2\left[\eta_0-3\xi_0'+(3\xi_0+2\eta_0')\nu\right]\sin\nu+$$
$$-2\left[12\xi_0+7\eta_0'+2+\xi_0'\nu\right]\cos\nu \quad (45b)$$

In fact, it has been confirmed by substitution that these solutions satisfy the system of Eqs. (19) to first order.

Second-order terms cannot easily be calculated and are in fact not useful as more accurate exact solutions have already been given.

By means of the well-known classical zeroth-order and the new first-order solutions established here, the relative motion of the probe under constant circumferential thrust can readily be expressed in a conventional perturbation series. The zeroth-order "unforced" Clohessy-Wiltshire solutions are well known (see for instance Ref. 6). [In fact, they will be given below in Eqs. (48)].

Finally, the special solutions for the case when all initial conditions are zero will be presented. These results can be used for visualization of the evolution of an orbit under circumferential thrust relative to the corresponding unperturbed circular orbit

$$\xi(\nu)=2\epsilon(\nu-\sin\nu)+\mathcal{O}(\epsilon^2) \quad (46a)$$

$$\eta(\nu)=4\epsilon(1-\cos\nu)-3\epsilon\nu^2/2+\mathcal{O}(\epsilon^2) \quad (46b)$$

B. Radial Thrust

It is of interest to obtain first-order approximate solutions of the system presented in Eqs. (21). Thereto, expansions of the constants ω_1, ω_2, C_1, S_1, C_2, S_2 appearing in Eqs. (37) and (38) are required

$$\omega_1=1-2\epsilon+\mathcal{O}(\epsilon^2) \quad (47a)$$

$$\omega_2=\sqrt{3\epsilon}\left\{1+2\epsilon+\mathcal{O}(\epsilon^2)\right\} \quad (47b)$$

$$C_1=-3\xi_0-2\eta_0'-\epsilon(1+24\xi_0+14\eta_0')+\mathcal{O}(\epsilon^2) \quad (47c)$$

$$S_1=\xi_0'+2\epsilon(3\xi_0'-\eta_0)+\mathcal{O}(\epsilon^2) \quad (47d)$$

$$C_2=4\xi_0+2\eta_0'+\epsilon(24\xi_0+14\eta_0'+4/3)+\mathcal{O}(\epsilon^2) \quad (47e)$$

$$S_2=2\sqrt{\epsilon/3}\left\{\eta_0-2\xi_0'+\epsilon(5\eta_0-16\xi_0')+\mathcal{O}(\epsilon^2)\right\} \quad (47f)$$

It is seen that ω_2 and S_2 contain terms of order $\sqrt{\epsilon}$. These square roots will cancel in the final results because of multiplication with similar terms.

After expansion of the trigonometric functions in Eqs. (36), the zeroth-order results can be written as

$$\xi^{(0)}(\nu)=\xi_0+(3\xi_0+2\eta_0')(1-\cos\nu)+\xi_0'\sin\nu \quad (48a)$$

$$\eta^{(0)}(\nu)=\eta_0-3(2\xi_0+\eta_0')\nu-2\xi_0'(1-\cos\nu)$$
$$+2(3\xi_0+2\eta_0')\sin\nu \quad (48b)$$

These results are identical to the ones for the uncontrolled Clohessy-Wiltshire equations (which provide a useful check).

The first-order results can eventually be rearranged in the following form:

$$\xi^{(1)}(\nu)=(1+24\xi_0+14\eta_0')(1-\cos\nu)+\xi_0'\sin\nu-2\xi_0'\nu\cos\nu$$
$$-2(3\xi_0'+2\eta_0')\nu\sin\nu+2(\eta_0'-2\xi_0')\nu-3(2\xi_0+\eta_0')\nu^2 \quad (49a)$$

$$\eta^{(1)}(\nu)=(4\eta_0-14\xi_0')(1-\cos\nu)+2(1+27\xi_0+16\eta_0')\sin\nu$$
$$+4\xi_0'\nu\sin\nu-4(3\xi_0+2\eta_0')\nu\cos\nu-2(1+21\xi_0+12\eta_0')\nu$$
$$-3(\eta_0-2\xi_0')\nu^2/2+(2\xi_0+\eta_0')\nu^3 \quad (49b)$$

The complete solution to first-order accuracy is now readily obtained by addition of the results of Eqs. (48) and (49).

It is of interest to calculate the special results for the case when all initial conditions are zero. This particular case describes the motion of the satellite under constant radial thrust relative to the corresponding free orbit:

$$\xi(\nu)=\epsilon(1-\cos\nu)+\mathcal{O}(\epsilon^2) \quad (50a)$$

$$\eta(\nu)=-2\epsilon(\nu-\sin\nu)+\mathcal{O}(\epsilon^2) \quad (50b)$$

VI. Application of the Results

In many applications, fuel-optimum rendezvous trajectories need to be determined under the assumption that the propulsion system of the probe is capable of providing constant and finite thrust along the circumferential and radial directions. Since normally the thrusting periods extend over a finite time, the maneuvers cannot be considered as impulsive, and questions on the optimal duration and sequence of thrust arcs arise.

The problem to be considered here consists of determining the circumferential thrust history for the fuel-optimum rendezvous maneuvering between the initial probe orbit and the final target orbit, where both orbits are considered to be circular and coplanar.

A. Numerical Algorithm

For the solution of this optimization problem, the standard recursive quadratic numerical algorithm OPXRQP[7] has been used. The optimization is carried out in the inertial geocentric reference frame, i.e., on the basis of Eqs. (7). The parameters to be optimized for each thrust and coast arc are the start and end times of the thrust intervals. The performance index is the mass remaining after completion of the rendezvous. Constraints arise from the condition that at the applicable time, the achieved orbital elements including the orbital phase must match those of the target orbit within predefined tolerances. An initial coast arc is allowed so that rendezvous can be achieved using only two finite-thrust burns (as for the corresponding impulsive Hohmann transfer maneuvers).

To start the optimization process, initial guesses of the variables to be optimized are required, and these are usually derived from the approximate impulsive solutions provided by the Clohessy-Wiltshire model. The corresponding finite-thrust intervals are constructed with midpoints coinciding with the impulsive thrusts. The optimization algorithm integrates the equations of motion on the basis of this set of initial guesses. At the end of the last thrust arc, the achieved orbit together with the total fuel used and the values of the constraints are calculated. The algorithm proceeds iteratively by determining the direction in which the optimization criterion, i.e., the final mass, is improved and/or the violation of the constraints is reduced until no further significant improvement in these values can be obtained. After the optimal maneuver parameters have been determined, the time history of the relative position and velocity of the probe with respect to the target can be computed by subtracting the characteristics of the two orbits.

B. Use of the Analytical Results

The rendezvous trajectories of the probe have also been obtained by substitution of the analytical results for circumferential thrust established previously into the numerical optimization scheme. These formulas provide the position and velocity of the probe relative to the target under the assumption that a constant thrust force is acting on the probe over the different thrust (including coast) arcs. The results obtained with this "analytical" procedure have been compared with those resulting from the completely numerical algorithm for a practical rendezvous scenario.

C. EURECA Retrieval Example

In the example considered, the probe and target orbits are assumed to be coplanar, and the probe is allowed to coast in the initial orbit until the appropriate phase angle for starting the rendezvous maneuver is established. The example is based on a realistic scenario related to the retrieval maneuvers of the EURECA spacecraft with the Shuttle. The target (Shuttle) orbit is taken as circular at 315 km altitude. The acceleration delivered by EURECA thrusters is $0.0206\ m/s^2$ assuming a mass of 3400 kg and a thrust level of 70 N.

Figure 3 illustrates the trajectory of EURECA relative to the target during the rendezvous maneuver obtained with the numerical procedure (continuous line) and with the exact circumferential solution (dashed line). On the scale of Fig. 3, the approximate solution coincides with the exact solution. The initial semimajor axis difference is $+10$ km, and the phase angle difference is $+5$ deg, which amounts to a range of about 580 km. It should be mentioned that an extended coast arc precedes the first thrust arc, which starts at a range of about 27 km. The crosses shown in each of the trajectories represent time increments of 6 min. The closer spacing at the end of the maneuver reflects the smaller relative velocity.

When the exact analytical results are used, the error at rendezvous amounts to 291 m in the x component and 818 m in the y component, see Table 1. The use of the approximate analytical solution, on the other hand, results in a slightly worse error in the y component with respect to the exact solution of only about 20 m. This is a small error in comparison with the error of both analytical formulations with respect to the numerical results. These results indicate therefore that the errors of both analytical solutions are introduced because the relative motion model used in the analysis has been linearized in terms of the ratio of the relative vs the orbital distance. This intro-

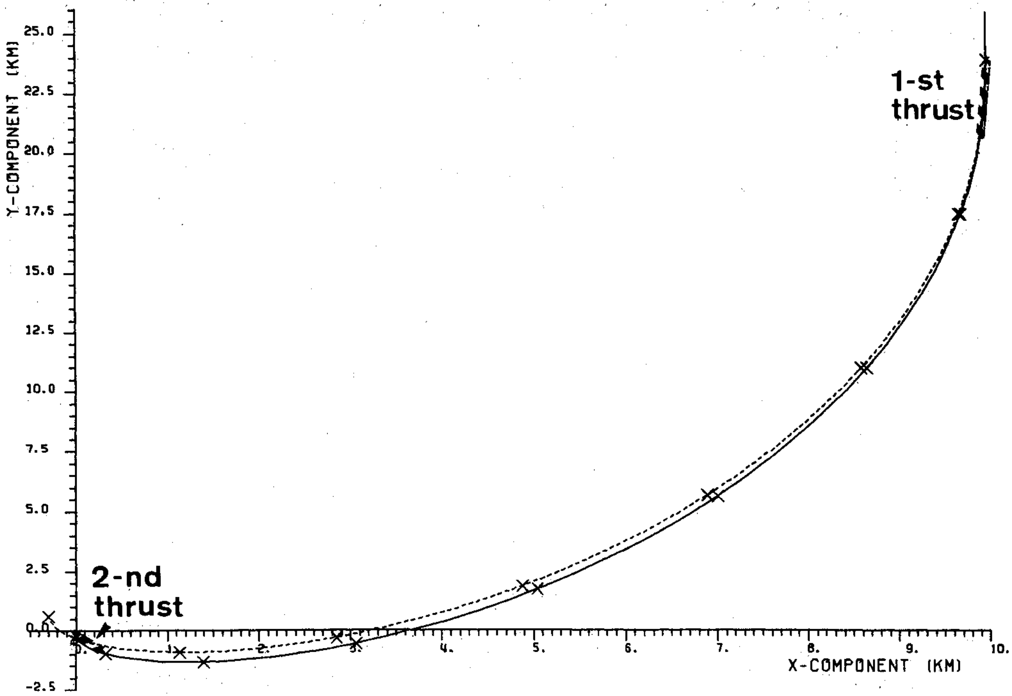

Fig. 3 Comparison of analytical and numerical rendezvous trajectories.

Table 1 Accuracies of circumferential thrust solutions in meters

	End 1st thrust arc, 2.3 min	End coast arc, 43 min	End 2nd thrust arc, 2.3 min
	Exact analytical results		
x comp	4	293	291
y comp	1	750	818
	Approximate analytical results		
x comp	5	294	292
y comp	1	758	839

Table 2 Accuracies of radial thrust solutions

Exact analytical results		
Range, km	Error in x comp, m	Error in y comp, m
25	7	1
50	45	9
350	534	284

Table 3 Accuracy of approximate analytical solution

Approximate analytical results			
Thrust level, N	Value of $\epsilon \times 10^3$	Error in x comp, m	Error in y comp, m
10	0.3	0.002	0
45	1	0.2	0.02
70	2	0.7	0.08
140	5	6	0.8
280	9	40	7
560	20	400	60

duces increasing errors for growing relative distances. The error due to the perturbation expansion in ϵ in the approximate solution is relatively insignificant in comparison.

A complete summary of the resulting accuracies of the analytical models (with respect to the numerical integration) at the various stages of the relative trajectory is provided in Table 1. The figures shown are the final relative position errors in meters for the rendezvous maneuver starting out with initial values $\Delta a = 10$ km and $\Delta\nu = 5$ deg.

In conclusion, the comparison between the numerical and the analytical results shows that the two analytical methods can be used for representing the relative motion of the probe under circumferential thrusting, provided that the range between the two vehicles remains relatively short.

It may be mentioned that there are techniques for improving the validity of the relative motion solutions over large ranges, namely by "curving" the y-coordinate of the local station frame along with the gravity field line, see for instance Ref. 8. It is expected that, with such a technique, the analytical solutions can be significantly improved for relatively large initial ranges.

D. General Accuracy Assessment

The results for radial thrust are summarized in Table 2. The figures represent the final relative position errors in meters after a thrust duration of about 5 min. The same thrust value as used above has been taken here. The comparison is based on the results of the exact analytical formulation relative to numerical integration over the thrust arc.

As before, the errors in the final relative position obtained by the analytical formulation are caused by the linear nature of the theory. In comparison, the errors induced by the approximation of the thrust expansion are much less significant.

Finally, the numerical and analytical results have been compared also for different thrust values under circumferential and radial thrusting and for different combinations of the initial locations of the probe and the target. The results of the comparison between the exact and approximate analytical theories for different ϵ values for the case of circumferential thrusting are presented here. This evaluation provides an indication of the acceptability of the approximate analytical solution for different values of the perturbation parameter. The comparisons are made after the first thrust arcs of a fixed 2.3 min duration.

Table 3 summarizes the results which show that significant errors occur only for ϵ values larger than about 0.01. The figures shown refer to the final relative position errors in meters. It should be pointed out that the thrust levels shown are calculated from the ϵ values by using the EURECA mass value, i.e., 3400 kg. The initial range difference is taken as 584 km, but the thrust starts at a range of about 27 km.

VII. Concluding Remarks

A general model for relative motion under arbitrary forces on each of the two satellites has been constructed and applied in a realistic rendezvous scenario. The major results of the paper may be summarized as follows.

1) A general formulation for the relative motion under arbitrary forces on each of the satellites has been established.

2) Exact analytical results for the linear Clohessy-Wiltshire equations including constant circumferential or radial thrust have been derived.

3) From the exact solutions, first-order (in terms of thrust/gravity forces) perturbation solutions have been obtained by expansion.

4) The analytical results have been imbedded in an optimization algorithm used in rendezvous problems and applied to a realistic scenario involving the European Space Agency's retrievable carrier (EURECA) and the Shuttle.

5) The accuracies of the analytical results were assessed by comparison with those of a numerical algorithm. It is concluded that the errors are acceptable for many practical applications.

6) The error induced by the linearization in terms of the relative range dominates the error due to expansion of the thrust parameter performed in the approximate analytical theory. Therefore, there is no significant loss of accuracy when using the approximate analytical results.

7) The analytical results obtained here should be useful in mission analysis of rendezvous problems as well as for potential onboard implementations.

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