# Non-Singular and Non-Conventional Orbit Perturbation Equations

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*Summary: A few attractive systems of equations describing perturbed orbital motion are derived from first principles. These equations are particularly useful for near-circular and near-equatorial satellite orbits where the classical perturbation equations exhibit singularities. Instrumental in the derivation of the results is the use of a quasi-angle defined by a differential relation. This leads to a natural decoupling of in-plane and out-of-plane perturbing effects and to relatively compact expressions. Finally, a few convenient non-conventional orbital elements which have a non-vanishing rate of change in the absence of perturbations are presented.*

**Nichtsinguläre und nichtkonventionelle Störungsgleichungen in der Bahnmechanik**

*Übersicht: Es werden einige neue Beschreibungen von beliebig gestörten Satellitenbahnen mitgeteilt. Diese sind für Bahnen mit geringer Exzentrizität und Bahnneigung von praktischer Bedeutung. In diesen Fällen werden die klassischen Differentialgleichungen aufgrund von Singularitäten unbrauchbar. Die Ergebnisse werden durch die Entkopplung von Störungseffekten, welche auf die Bahninklination einwirken, und solche, die nur innerhalb dieser Ebene wirksam sind, abgeleitet. Außerdem werden nichtkonventionelle Bahnelemente eingeführt, die dadurch gekennzeichnet sind, daß sie sich dauernd als Funktion der Zeit ändern, auch wenn alle Störungskräfte außer acht gelassen werden.*

## 1. Introduction

The well-known classical *Lagrange*'s Planetary Equations [1] describe the rate of change of a set of orbital elements under a combination of arbitrary perturbing forces. In the conventional terminology the orbital elements would become constants if the perturbing forces were to vanish: such variables may be called slow orbital elements. On the other hand, there also exist orbital elements with a non-vanishing rate of change in the absence of perturbations, e.g. the mean and true anomalies. These variables are designated as fast elements. It is evident that in the numerical integration of the set of first-order differential equations for the variation of the elements, a fast variable would generally require a much smaller step size than the one needed for a slow element, if consistent accuracies are

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desired. The ratio of the stepsizes would be $\varepsilon$ to 1, where $\varepsilon$ denotes the order of magnitude of the perturbing forces after proper nondimensionalization. This observation does not imply, however, that fast orbital elements are in all cases to be disregarded as possible candidates for describing perturbed orbital motion. In the present paper a few situations are discussed where fast orbital elements can be employed without adverse consequences on the step size in the numerical integration process. The usefulness of fast elements is based on the following considerations:

1. Any arbitrary fast orbital element defines a corresponding slow variable if its unperturbed *Keplerian* contribution is simply subtracted. These so-called slow difference-elements would vanish identically in the case of unperturbed motion.

2. In particular cases, a normally fast element may be considered as a slow variable. If, for example, the eccentricity $e$ can be proven to remain of the same order of smallness as the perturbation parameter $\varepsilon$ throughout the time-interval of interest, the rate of change of the fast element $e \cos \vartheta$ (where $\vartheta$ is the "fast" true anomaly) would effectively be a slow element in the sense that its rate of change is of similar magnitude as that of a conventional slow element. It should be kept in mind, however, that $e \cos \vartheta$ is not slow in the sense that it would become a constant in the absence of perturbations. Therefore, such an element will be called quasi-slow, where it must be recognized that the slowness holds only for particular initial conditions and for particular perturbing forces.

The objective of investigating these quasi-slow orbital elements is to arrive at more convenient representations for the well-known *Lagrange* Planetary Equations describing the variation of the classical orbital elements $a$, $e$, $\omega$, $i$, $\Omega$ and $T$. The *Gauss* formulation, in particular, where the perturbing forces are expanded in three local components along radial, transverse and orbit-normal directions results in quite awkward equations which do not readily admit an analytical approach for any non-trivial application. In numerical integration the advantage of a more compact set of equations in terms of non-conventional elements would lie in a faster execution time. It may be mentioned that although a canonical set of perturbation equations would be more compact than any other set a price in terms of fairly elaborate transformations for the actual physical perturbing forces into canonical forces must be paid for this convenience [2]. Compactness and ease of application of the equations as well as the complexity of the perturbing force expressions therefore play a role in assessing the suitability of a particular formulation. It should also be emphasized that different applications may lead to different preferred forms.

The ideas presented in this paper has been suggested by investigations related to the selection of a suitable formulation for relative motion problems [3]. The use of a pseudo-angle $\nu$ defined by the differential relation $\dot{\nu} = h/r^2$ in perturbed orbital motion has been advocated by many authors [4 to 7]. This choice is instrumental in eliminating out-of-plane perturbing force components from the in-plane perturbation equations which is an essential advantage (especially in analytical work) when compared to conventional formulations.

## 2. Nomenclature

| | |
|---|---|
| $\mathbf{a}$ | (general) array of orbital elements |
| $a$ | semi-major axis |
| $D$ | auxiliary variable, Eqs. (31) |
| $e$ | eccentricity |
| $f, g$ | non-singular orbital elements, Eqs. (37) |
| $\mathbf{f}$ | acceleration vector |
| $\mathbf{h}$ | angular momentum (per unit mass) vector |
| $i$ | inclination |
| $I, J, K$ | non-singular orbital elements, Eqs. (41) |
| $J_2$ | second zonal harmonic of Earth's potential field |
| $j, k$ | non-singular orbital elements, Eqs. (28) |
| $p, q$ | cartesian coordinates of eccentricity vector, Eqs. (22) |
| $\mathbf{r}$ | position vector |
| $R_e$ | Earth's radius |
| $T$ | time of perigee passage |
| $\mathbf{u}_x, \mathbf{u}_y, \mathbf{u}_z$ | unit-vectors along $x$, $y$, $z$ axes, FIG. 1 |
| $\mathbf{v}$ | velocity vector, $\dot{\mathbf{r}}$ |
| $\mathbf{w}$ | rotation vector of local frame in inertial frame |
| $\delta$ | (general) difference element |
| $\varepsilon$ | (general) small parameter |
| $\vartheta$ | true anomaly |
| $\nu$ | quasi-angle, Eqs. (19) |
| $\mu$ | central body's gravitational parameter |
| $\sigma$ | auxiliary element, Eq. (30) |
| $\varphi$ | argument of latitude, $\vartheta + \omega$ |
| $\psi$ | quasi-angle, Eqs. (19) |
| $\omega$ | argument of perigee |
| $\tilde{\omega}$ | modified argument of perigee, Eqs. (21) |
| $\Omega$ | right ascension of ascending node |

Superscripts $\dot{}$ and $'$ refer to differentiation with respect to time and quasi-angle $\nu$, respectively.

## 3. Desirable Properties of Perturbation Equations

The evolution of a satellite orbit under perturbing forces is described by a system of at least six differential equations representing the rate of change of a set of variables describing the orbital state.

Symbolically, it is possible to write:

$$\text{(1)} \qquad \dot{\mathbf{a}} = \mathbf{F}\,[\mathbf{a}, \mathbf{f}\,(t)], \quad \mathbf{a}\,(0) = \mathbf{a}_0,$$

where the three-vector $\mathbf{f}$ represents the small perturbing accelerations. There is quite an amount of liberty in the choice of a suitable set of state variables or elements $\mathbf{a}$ describing the perturbed motion. Depending on the application in mind one set of elements could be superior to another set. Although it would be futile to try to develop unambiguous criteria on the basis of which the best set of elements could be selected in all possible applications, it is certainly worthwhile to identify a few desirable features serving as a basis for the selection of a suitable set for a given application.

A few criteria to be taken as guidelines for the selection of a suitable set of orbital parameters would be the following:

a) The perturbation equation (1) should be non-singular in terms of the set of elements **a**. This means that there exists a uniform upper bound $M$ on the absolute value of **F** for all possible values of **a** and **f**:

(2) $\quad |\mathbf{F}\,(\mathbf{a},\mathbf{f})| < M.$

b) The elements should be slowly varying, i.e. they should become constants in the absence of perturbations:

(3) $\quad \mathbf{F}\,(\mathbf{a},\mathbf{o}) = \mathbf{o} \Rightarrow \mathbf{a}\,(t) \equiv \mathbf{a}_0.$

This property would be advantageous in the numerical integration of the perturbation equations because it permits the use of a large step size.

Since it is felt that the latter criterion is too restrictive, the following more liberal criterion may be adopted:

b) The elements are allowed to be quasi-slowly varying, i.e. the elements should remain of the order of $\varepsilon$ ($\varepsilon$ representing the order of smallness of the perturbations) during unperturbed motion:

(4) $\quad |\mathbf{F}\,(\mathbf{a},\mathbf{o})| = O\,(\varepsilon).$

This relaxation does not affect the step size required for numerical integration and opens up new avenues for the selection of suitable orbit parameters.

c) It is desirable that the chosen set of elements leads to an uncoupling of the out-of-plane perturbing force components from the in-plane motion. This property is of particular importance in analytical perturbation analyses.

d) In order to minimize computation time, it is desirable that the equations should be as "compact" as possible. On the one hand, this sets a limit on the number of equations ($\geq 6$) required to describe the complete system and, on the other hand, the number of numerical operations within each equation should be minimized.

e) The perturbing force components should have a convenient form when expressed in terms of the selected set of elements.

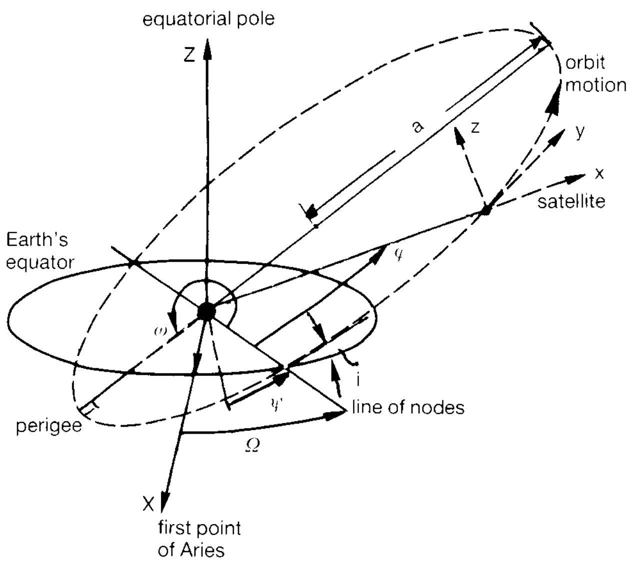


FIG. 1: Visualization of perturbed orbital motion

## 4. Orbit Mechanics Background

The evolution of a satellite's position vector $\mathbf{r}\,(t)$ within an inertial refence frame is described by *Newton*'s second law:

(5) $\quad \ddot{\mathbf{r}} = -\mu\,\mathbf{r}/r^3 + \mathbf{f}.$

In addition to the inertial geocentric $X, Y, Z$ reference frame a local moving $x, y, z$ frame is introduced (FIG. 1). In this, the $x$ axis points along the instantaneous position vector **r** while the $y$ axis lies in the plane defined by the instantaneous position and velocity vectors **r** and **v** such that **v** has a positive $y$ component. The plane containing the **r** and **v** vectors is known as the osculating plane and changes its orientation continually under the influence of the normal perturbing force component. The $z$ axis is directed along the instantaneous normal to the osculating plane and thus points along the vector $\mathbf{h} = \mathbf{r} \times \mathbf{v}$.

Similarly, as in rigid body dynamics, the motion of the local reference frame with respect to the inertial frame may be described by means of a rotation vector **w**. The rate of change of any vector quantity **b** can now be expressed as $\dot{\mathbf{b}} = [\dot{\mathbf{b}}] + \mathbf{w} \times \mathbf{b}$ where $[\dot{\mathbf{b}}]$ refers to the derivative of **b** evaluated in the local frame. Application of this rule to the vector **r** yields:

(6) $\quad \mathbf{v} = \dot{\mathbf{r}} = \dot{r}\,\mathbf{u}_x + r\,w_z\,\mathbf{u}_y - r\,w_y\,\mathbf{u}_z.$

From the definition of the osculating plane, the velocity vector cannot have a component along the $z$ axis, so that

(7) $\quad w_y = 0.$

Differentiation of the velocity vector in the same manner, taking account of the result in Eq. (7), leads to:

(8)
$$\ddot{\mathbf{r}} = \ddot{r}\,\mathbf{u}_x + (r\,w_z)^{\cdot}\,\mathbf{u}_y - r\,w_z^2\,\mathbf{u}_x + \dot{r}\,w_z\,\mathbf{u}_y + r\,w_x\,w_z\,\mathbf{u}_z.$$

On the other hand, the right-hand-side of Eq. (4) can be expanded along the local axes to give:

(9) $\quad \ddot{\mathbf{r}} = (f_x - \mu/r^2)\,\mathbf{u}_x + f_y\,\mathbf{u}_y + f_z\,\mathbf{u}_z.$

Comparing the components of $\ddot{\mathbf{r}}$ in Eqs. (8) and (9), the following three identities are found:

(10)
$$\ddot{r} - r\,w_z^2 = f_x - \mu/r^2,$$
$$r\,\dot{w}_z + 2\,\dot{r}\,w_z = f_y,$$
$$r\,w_x\,w_z = f_z.$$

Explicit expressions for $w_x$ and $w_z$ can be obtained by studying the rate of change of the angular momentum vector **h**:

$$(11) \quad \dot{\mathbf{h}} = \mathbf{r} \times \mathbf{f} = r\,(f_y\,\mathbf{u}_z - f_z\,\mathbf{u}_y)\,.$$

Because of the differentiation rule introduced above, $\dot{\mathbf{h}}$ can also be expressed as:

$$(12) \quad \dot{\mathbf{h}} = \dot{h}\,\mathbf{u}_z + h\,(w_y\,\mathbf{u}_x - w_x\,\mathbf{u}_y)\,.$$

Comparing the results of Eqs. (11) and (12), the following expressions are found in addition to the already known result of Eq. (7):

$$(13) \quad \dot{h} = r\,f_y;\; w_x = r\,f_z/h\,.$$

The last relationship of Eqs. (10) provides the remaining component of **w**:

$$(14) \quad w_z = h/r^2\,.$$

The results obtained so far may be summarized as:

$$(15) \quad \ddot{r} - h^2/r^3 = f_x - \mu/r^2;\; \dot{h} = r\,f_y;$$

$$(16) \quad w_x = r\,f_z/h;\; w_y = 0;\; w_z = h/r^2\,.$$

The first set, i.e. Eqs. (15), describes the in-plane perturbed motion whereas Eqs. (16) represent the rotation of the local $x, y, z$ frame attached to the instantaneous **r** and $\dot{\mathbf{r}}$ vectors relative to inertial space, FIG. 1. Eqs. (15) and (16) together form a convenient starting point for the derivation of various sets of perturbation equations in the *Gauss* form.

The motion of the local $x, y, z$ frame can readily be interpreted as that of a rigid body spinning around the $z$-axis with a spin rate $w_z$ and perturbed by a disturbance torque $r f_z$, resulting in an instantaneous rotation rate $w_x$.

## 5. Non-Singular Perturbation Equations

The conventional *Lagrange* Planetary Equations [1] are not the most suitable perturbation equations for many practical applications. The singularities for small eccentricities and small inclination in the equations for $\dot{\omega}$ and $\dot{\Omega}$, respectively, may lead to difficulties in the integration. Furthermore, the osculating (true, mean, and eccentric) anomalies become ill-defined for small eccentricity.

Before presenting a non-singular alternative to the *Lagrange* Planetary Equations the motion of the local $x, y, z$ reference frame is considered in more detail. The orientation of this plane is usually described by the "*Euler* angles" $\Omega$, $i$, $\varphi$ (FIG. 1). The components of the rotation vector **w** can thus be expressed in terms of $(i)^{\cdot}$, $\dot{\Omega}$ and $\dot{\varphi}$:

$$(17) \quad \begin{aligned} w_x &= (i)^{\cdot}\,\cos\varphi + \dot{\Omega}\,\sin i\,\sin\varphi\,, \\ w_y &= -\,(i)^{\cdot}\,\sin\varphi + \dot{\Omega}\,\sin i\,\cos\varphi\,, \\ w_z &= \dot{\varphi} + \dot{\Omega}\,\cos i\,. \end{aligned}$$

Using the results in Eqs. (16), these can be solved for:

$$(18) \quad \begin{aligned} (i)^{\cdot} &= (r\,f_z/h)\,\cos\varphi\,, \\ \dot{\Omega} &= (r\,f_z/h)\,\sin\varphi/\sin i\,, \\ \dot{\varphi} &= h/r^2 - \dot{\Omega}\,\cos i\,. \end{aligned}$$

Thus, Eqs. (18) are equivalent to Eqs. (16) and, together with Eqs. (15), provide a complete description of the perturbed orbital motion.

### 5.1. Equations for Near-Circular Orbits

An essential step for the following analysis is to define the quasi-angles $\nu$ and $\psi$, using the differential equations:

$$(19) \quad \dot{\nu} = w_z = h/r^2;\; \dot{\psi} = \dot{\Omega}\,\cos i\,.$$

These definitions permit formal integration of the last relationship in Eqs. (18):

$$(20) \quad \nu = \varphi + \psi\,.$$

In the absence of perturbations, $\psi$ may be taken to be zero and $\nu$ becomes equal to the argument of latitude $\varphi = \omega + \vartheta$ with constant $\omega$. In the perturbed case, $\nu$ may be expressed in terms of the osculating true anomaly by introducing a "modified" argument of perigee $\tilde{\omega}$:

$$(21) \quad \nu = \vartheta + \tilde{\omega};\quad \tilde{\omega} = \omega + \psi\,.$$

The radial position of the satellite can thus be written as a function of $\nu$:

$$\begin{aligned} r\,(\nu) &= (h^2/\mu)/(1 + e\,\cos\vartheta) = \\ &= (h^2/\mu)/(1 + p\,\cos\nu + q\,\sin\nu)\,, \end{aligned}$$

where

$$(22) \quad p = e\,\cos\tilde{\omega};\; q = e\,\sin\tilde{\omega}\,.$$

The radial component $\dot{r}$ of the velocity vector is equal to:

$$(23) \quad \dot{r} = \dot{\nu}\,dr/d\nu = (\mu/h)\,(p\,\sin\nu - q\,\cos\nu)\,.$$

Here, use is made of the fact that in arbitrary perturbed motion, the position and velocity vectors have the same form in terms of the instantaneous elements as in the unperturbed case (i.e. condition of osculation). This implies that the following constraint equation needs to be satisfied:

$$(24) \quad \dot{p}\,\cos\nu + \dot{q}\,\sin\nu = 2\,h\,\dot{h}/(\mu\,r) = 2\,h\,f_y/\mu\,,$$

where the last of Eqs. (15) has been substituted. A second equation for $\dot{p}$ and $\dot{q}$ follows from the equation for $\ddot{r}$ in Eqs. (15) by differentiating Eq. (23):

$$(25) \quad \begin{aligned} &\dot{p}\,\sin\nu - \dot{q}\,\cos\nu = \\ &\qquad = h\,f_x/\mu + r\,f_y\,(p\,\sin\nu - q\,\cos\nu)/h\,. \end{aligned}$$

Eqs. (24) and (25) readily yield the desired perturbation equations for $p$ and $q$:

$$(26)\quad \begin{aligned} \dot{p} &= (h/\mu)\,\{f_x \sin\nu + \\ &\quad + f_y\,[\cos\nu + \mu\, r\,(p + \cos\nu)/h^2]\}, \\ \dot{q} &= (h/\mu)\,\{-f_x \cos\nu + \\ &\quad + f_y\,[\sin\nu + \mu\, r\,(q + \sin\nu)/h^2]\}. \end{aligned}$$

It is important to recognize that Eqs. (26) do not contain $f_z$, i.e. the normal (out-of-plane) perturbing force component. This is a clear advantage over the conventional equinoctial elements [8], especially for analytical work. Also it should be appreciated that this formulation results in generally more compact expressions than more conventional formulations of the *Gauss'* form.

The complete set of perturbation equations for near-circular orbits may now be summarized as:

$$(27)\quad \begin{aligned} \dot{h} &= r\,f_y, \\ \dot{p} &= (h/\mu)\,\{f_x \sin\nu + \\ &\quad + f_y\,[\cos\nu + \mu\, r\,(p + \cos\nu)/h^2]\}, \\ \dot{q} &= (h/\mu)\,\{-f_x \cos\nu + \\ &\quad + f_y\,[\sin\nu + \mu\, r\,(q + \sin\nu)/h^2]\}, \\ (i)^{\cdot} &= (r\,f_z/h)\cos(\nu - \psi), \\ \dot{\psi} &= (r\,f_z/h)\sin(\nu - \psi)\cot i, \\ \dot{\nu} &= h/r^2, \\ (\dot{\Omega} &= \dot{\psi}/\cos i). \end{aligned}$$

It can be argued that the angle $\psi$ makes the equation for $\Omega$ superfluous. For a geometric description of the orbit plane motion, however, the angle $\Omega$ offers definite advantages over $\psi$ and may therefore be included as an extra equation.

Finally, it should be stressed that the perturbing force components $f_x, f_y, f_z$ need to be expressed in the set of elements adopted. This involves a great deal of labor for realistic perturbations but should be no more difficuld in the present formulation than in other non-singular theories (e.g. equinoctial).

### 5.2. Equations for Near-Equatorial Orbits

For orbits with inclination near 0 or $\pi$ (referred to as near-equatorial) a singularity in the equations for $\Omega$ and $\psi$ may appear, cf. Eqs. (18). A non-singular formulation for the out-of-plane motion follows readily from the analysis above. Introducing the new variables:

$$(28)\quad j = \sin i \cos\psi;\; k = \sin i \sin\psi$$

Eqs. (18), (19) and (20) can be used to show that:

$$(29)\quad \begin{aligned} (j)^{\cdot} &= \pm\,(r\,f_z/h)\,(1 - j^2 - k^2)^{1/2}\cos\nu, \\ \dot{k} &= \pm\,(r\,f_z/h)\,(1 - j^2 - k^2)^{1/2}\sin\nu, \end{aligned}$$

where +, − holds for prograde (i.e. $i < 90°$) and retrograde ($i > 90°$) orbits. The two equations for $j$ and $k$ in Eqs. (29) replace those for $i$ and $\psi$ in Eqs. (27). For a near-equatorial orbit the ascending node position becomes ill-defined: small perturbing influences may shift it by large amounts.

Unlike the behavior of $\Omega$ the new variable:

$$(30)\quad \sigma = \Omega \mp \psi$$

remains well-defined for prograde and retrograde near-equatorial orbits so that the following equation may be added:

$$(31)\quad \begin{aligned} \dot{\sigma} &= D\,(r\,f_z/h)\,(j\sin\nu - k\cos\nu); \\ D &= 1/[1 + (1 - j^2 - k^2)^{1/2}]. \end{aligned}$$

By integrating Eq. (31) using $\sigma(0) = \Omega_0$ along with Eqs. (29), $\Omega$ may be obtained as:

$$(32)\quad \Omega = \sigma \pm \psi = \sigma \pm \arctan(k/j),$$

without loss of precision for near-equatorial orbits. After adding the equations for $h, p, q$ and $\nu$ given in Eqs. (27), a system of perturbation equations which is non-singular for both near-circular and near-equatorial orbits has been obtained.

For applications where the orbit is eccentric with a near-equatorial orbital plane, the equations for $p, q$ may be replaced by a more familiar form for $e$ and $\tilde{\omega} = \omega + \psi$:

$$\begin{aligned} \dot{e} &= (h/\mu)\,\{f_x \sin\vartheta + \\ &\quad + f_y\,[\cos\vartheta + \mu\, r\,(e + \cos\vartheta)/h^2]\}, \\ \dot{\tilde{\omega}} &= [h/(\mu\, e)]\,\{-f_x\cos\vartheta + \\ &\quad + f_y\,(1 + \mu\, r/h^2)\sin\vartheta\}, \end{aligned}$$

where

$$(33)\quad \vartheta = \nu - \tilde{\omega};\; r = (h^2/\mu)/[1 + e + \cos(\nu - \tilde{\omega})].$$

The advantage of $\tilde{\omega}$, as compared with the usual argument of perigee $\omega$, is the absence of the normal perturbing force component in its perturbation equation. Thus, a decoupling of the in-orbit motion from the osculating plane "rigid-body" motion is accomplished. It is evident that the introduction of the quasi-angles $\nu$ and $\psi$ have been instrumental in this decoupling. The usefulness of these angles has already been recognized by *Hansen* whose ideal coordinate frame [4] is obtained from the local $x, y, z$ coordinate frame (FIG. 1) after a reverse rotation over an angle $\nu = \varphi + \psi$.

Another asset of the formulation in terms of $\nu$ is the fact that its perturbation equation $\dot{\nu} = h/r^2$ is more compact than the corresponding ones for any of the well-known anomalies, which all have terms containing the perturbing force components. This advantage will be appreciated when expressing the perturbation equation in terms of $\nu$ as the independent variable, which would be a first step in an analytical approach. Since $d/d\nu = (1/\dot{\nu})\, d/dt = (r^2/h)\, d/dt$, the following expressions for the near-circular near-equatorial case are obtained:

$$h'(\nu) = (r^3/h)\, f_y,$$

$$p'(\nu) = (r^2/\mu)\,\{f_x \sin\nu + f_y\,[\cos\nu + \mu\, r\,(p + \cos\nu)/h^2]\},$$

$$q'(\nu) = (r^2/\mu)\,\{-f_x \cos\nu + f_y\,[\sin\nu + \mu\, r\,(q + \sin\nu)/h^2]\},$$

(34)

$$j'(\nu) = \pm\,(r^3 f_z/h^2)\,(1 - j^2 - k^2)^{1/2} \cos\nu,$$

$$k'(\nu) = \pm\,(r^3 f_z/h^2)\,(1 - j^2 - k^2)^{1/2} \sin\nu,$$

$$t'(\nu) = r^2/h,$$

$$[\sigma'(\nu) = D\,(r^3 f_z/h^2)\,(j \sin\nu - k \cos\nu)].$$

This system is still exact and permits the expression of all the variables in terms of time in a parametric manner after integration.

### 5.3. Difference Elements

In the previous Section, two fast variables (i.e. parameters which have a non-zero rate of change in the absence of perturbations) appear, namely $\nu$ and $t$. These variables need smaller step sizes in a numerical integration routine than those required by the slow orbital elements. A simple way of avoiding fast variables is provided by the so-called difference elements, which are obtained by subtracting the corresponding "unperturbed" part from the fast variable. In the absence of perturbations these difference elements vanish and can thus be considered as slow variables. In the present application, the following expressions are introduced:

(35) $\delta_\nu = \nu - \nu_u;\ \delta_t = t - t_u$

where the subscript $u$ refers to the unperturbed element. The perturbation equations for $\delta_\nu$ and $\delta_t$ follow readily:

(36)

$$\dot{\delta}_\nu = h/r^2 - h_0/r_u^2 = (r_u^2\,\delta_h - 2\,h_0\,r_u\,\delta_r - h_0\,\delta_r^2)/(r_u\,r)^2,$$

$$\delta_t'(\nu) = (h_0\,\delta_r^2 + 2\,h_0\,r_u\,\delta_r - r_u^2\,\delta_h)/(h_0\,h),$$

where

$$r_u = (h_0^2/\mu)/(1 + p_0 \cos\nu + q_0 \sin\nu),$$

$$\delta_r = r - r_u,\ \delta_h = h - h_0.$$

In order to avoid the loss of precision inherent in the subtraction of two almost identical quantities, the difference $\delta_r$ should be calculated with care [3]. The same reason suggests integrating $\delta_h$, $\delta_p$ and $\delta_q$ rather than $h, p, q$ themselves, which only makes a difference in the choice of initial conditions. The unperturbed quantities $\nu_u$ and $t_u$ are obtained from *Kepler*'s equation – $t_u(\nu)$ in a direct manner and $\nu_u(t)$ indirectly by iteration of the inverse equation.

## 6. Non-Conventional Perturbation Equations

As was mentioned in Section 3, orbital parameters which are not constants in unperturbed motion but vary within a band of a similar magnitude to that produced by the perturbations are referred to as quasi-slow elements. These parameters appear to be just as suitable as the conventional orbital elements for describing the perturbed orbital motion in both numerical and analytical work.

### 6.1. Near-Circular Orbits

This idea is particularly useful for near-circular orbits where the variations of $f$ and $g$ in unperturbed motion would be of a similar order of magnitude as those due to the perturbing influences. Introducing the quasi-slow elements:

(37)

$$f = e \cos\vartheta = p \cos\nu + q \sin\nu,$$

$$g = e \sin\vartheta = p \sin\nu - q \cos\nu,$$

the non-singular perturbation equations introduced in the previous Section can be written as:

(38)

$$\dot{h} = r\, f_y,$$

$$\dot{f} = -\,g\,h/r^2 + 2\,h\,f_y/\mu,$$

$$\dot{g} = f\,h/r^2 + h\,f_x/\mu + g\,f_y\,r/h,$$

$$(j)^{\cdot} = \pm\,(r\,f_z/h)\,(1 - j^2 - k^2)^{1/2} \cos\nu,$$

$$\dot{k} = \pm\,(r\,f_z/h)\,(1 - j^2 - k^2)^{1/2} \sin\nu,$$

$$\dot{\nu} = h/r^2,$$

where $r = (h^2/\mu)/(1+f)$. The equation for $\sigma$ should be added if an explicit expression for $\Omega$ is desired, as in Eqs. (31) and (32). This set of equations is extremely convenient due to its compact form. A similar form with true longitude ($L = \vartheta + \omega + \Omega$) rather than $\nu$ describing the local frame motion has been used in the study of relative motion under air drag and oblateness perturbations [3].

As before $\nu$ may be replaced by its slow difference element. It is of interest to point out that by introducing the difference elements $\delta_f$ and $\delta_g$, a system which can also be used for larger values of eccentricity is obtained. The price paid is rather lengthy expressions, where $t$ is taken as the independent variable.

In terms of the independent variable $\nu$, Eqs. (38) become even more compact:

$$h'(\nu) = r^3 f_y/h,$$

$$f'(\nu) = -\,g + 2\,r^2 f_y/\mu,$$

$$g'(\nu) = f + r^2 f_x/\mu + g\,r^2 f_y/h^2,$$

(39) $j'(\nu) = K\,(r^3 f_z/h^2) \cos\nu,$

yet
$$k'(\nu) = K\,(r^3 f_z/h^2)\sin\nu,$$
(39)
$$t'(\nu) = r^2/h,$$
$$[K'(\nu) = -(r^3 f_z/h^2)\,(j\cos\nu + k\sin\nu)],$$

where $K = \cos i$ has been introducted in order to avoid the sign ambiguity but at the cost of an extra equation. The constraint $K^2 = 1 - j^2 - k^2$ may be useful for checking the consistency of the results after integration.

The system in Eqs. (39) can quite readily by extended for arbitrary values of eccentricity by means of the difference-elements $\delta_f$ and $\delta_g$:

(40)
$$\delta_f'(\nu) = -\delta_g + 2r^2 f_y/\mu,$$
$$\delta_g'(\nu) = \delta_f + r^2 f_x/\mu + (g_u + \delta_g)\,r^3 f_y/h^2,$$

where $g_u(\nu) = e_0 \sin(\nu - \bar{\omega}_0)$. This illustrates how non-conventional elements can be converted to orbital parameters with a zero rate of variation in the absence of perturbations.

### 6.2. Near-Equatorial Orbits

In the case of small inclinations, a similar approach to that given above for small eccentricities may be followed. Assuming that the rate of change of the inclination in unperturbed motion is of a similar magnitude as that induced by the perturbations, the following quasi-slow variables may be introduced:

(41)
$$I = \sin i \cos(\nu - \psi) = j\cos\nu + k\sin\nu,$$
$$J = \sin i \sin(\nu - \psi) = j\sin\nu - k\cos\nu.$$

These variables satisfy the system:

(42)
$$\dot{I} = -J\,h/r^2 + K\,r f_z/h,$$
$$\dot{J} = I\,h/r^2,$$

where $K = \cos i = \pm(1 - I^2 - J^2)^{1/2}$. The variables $I$, $J$, $K$ can be interpreted as the components of a unit-vector along the inertial $Z$-axis projected upon the local $x, y, z$ coordinate axes [9].

A particularly attractive set is obtained when using $\nu$ as the independent variable, cf. Eqs. (39):

$$h'(\nu) = r^3 f_y/h,$$
$$f'(\nu) = -g + 2r^2 f_y/\mu,$$
$$g'(\nu) = f + r^2 f_x/\mu + g\,r^3 f_y/h^2,$$
(43)
$$I'(\nu) = -J + K\,r^3 f_z/h^2,$$
$$J'(\nu) = I,$$
$$t'(\nu) = r^2/h,$$
$$(K'(\nu) = -I\,r^3 f_z/h^2).$$

Here again, the equation for $K$ is redundant. The system in Eqs. (43) is valid for near-circular near-equatorial orbits. The compactness of this system makes it attractive for both analytical and numerical applications.

The equation for $\sigma$ defined in Eq. (30) here becomes:

(44)
$$\sigma'(\nu) = J\,(r^3 f_z/h^2)/(1 \pm K).$$

The introduction of difference-elements $\delta_I$ and $\delta_J$ would extend the usefulness of this system to any value of inclination:

(45)
$$\delta_I' = -\delta_J + K\,r^3 f_z/h^2,$$
$$\delta_J' = \delta_I.$$

In conjunction with Eqs. (40) a compact system with general applicability would be obtained.

## 7. Short Illustration of the Theory

A short illustration of the applicability of the considerations presented above will be given here. More extensive applications can be found in the literature, e.g. [3 and 7] which employ a few of the guidelines advocated here.

The main perturbing term due to the Earth's potential field is the second zonal harmonic commonly denoted as $J_2$. The perturbing force components exhibit an extremely simple form when written in terms of the "elements" $I, J$ of Eqs. (41) and $K = \cos i$:

(46)
$$f_x = -\varepsilon\,(1 - 3J^2)/r^4,$$
$$f_y = -2\varepsilon\,I\,J/r^4,$$
$$f_z = -2\varepsilon\,J\,K/r^4,$$

where $\varepsilon = 3\mu J_2 R_e^2/2$. After substituting these components into the equations for $I$, $J$ and $K$ in Eqs. (43) the following convenient system is obtained for a circular orbit:

(47)
$$J''(\nu) + J = -\delta\,J\,K^2,$$
$$K'(\nu) = \delta J\,J'\,K,$$

where $\delta = 3J_2(R_e/a)^2$. The initial conditions are specified as:

(48)
$$J(\nu_0) = \sin i_0 \sin(\nu_0 - \psi_0),$$
$$I(\nu_0) = \sin i_0 \cos(\nu_0 - \psi_0),$$
$$K(\nu_0) = \cos i_0.$$

To a first-order approximation, the equation for $J(\nu)$ in Eqs. (47) can immediately be solved:

(49)
$$J(\nu) = \sin i_0 \sin[(1 + \delta K_0^2)^{1/2}\,\nu].$$

From this result the behavior of the line of nodes can readily be checked. On the basis of the definition of $J(\nu)$ in Eqs. (41), it follows that

$$(50) \quad \psi(\nu) = -(\delta/2)\,\nu \cos^2 i_0 .$$

Thus, the well known regression of the nodes at a rate of (cf. [4], § 6.3)

$$(51) \quad \Delta\Omega = -3\, J_2\, \pi\, (R_e/a)^2 \cos i_0$$

per revolution has been reproduced.

## 8. Concluding Remarks

A number of perturbation equations in terms of non-singular and non-conventional orbital parameters have been presented. By the use of the quasi-angle $\nu$, consistent separation of in-plane motion on one hand and the orbit plane motion on the other hand has been achieved. The resulting systems should be useful in both analytical and numerical applications of perturbed orbital motion because of their compact size relative to more conventional perturbation equations. The applicability of a particular formulation is largely determined by the ease with which the perturbing influences can be incorporated.

## References


[1] *P. M. Fitzpatrick:* Principles of Celestial Mechanics. Academic Press, New York 1970.

[2] *E. L. Stiefel* and *G. Scheifele:* Linear and Regularized Celestial Mechanics. Springer-Verlag, Berlin/Heidelberg/New York 1971.

[3] *J. C. Van der Ha:* Three-dimensional subsatellite motion under air drag and oblateness perturbations. Celestial Mechanics **26** (1982), pp. 285–309.

[4] *F. T. Geyling* and *H. R. Westermann:* Introduction to Orbital Mechanics. Addison-Wesley, Reading 1971.

[5] *J. L. Junkins* and *J. D. Turner:* On the analogy between orbital dynamics and rigid body dynamics. J. Astron. Sci. **27** (1979), pp. 345–358.

[6] *A. A. Kamel:* New non-singular forms of perturbed satellite equations of motion. J. Guidance, Control, and Dynamics **6** (1983), pp. 387–392.

[7] *J. C. Van der Ha* and *V. J. Modi:* Analytical evaluation of solar radiation induced orbital perturbations of space structures. J. Astron. Sci. **25** (1977), pp. 283–306.

[8] *R. A. Broucke* and *P. Cefola:* On the equinoctial orbit elements. Celestial Mechanics **5** (1972), pp. 303–310.

[9] *Y. M. Kopnin:* A rotation of the orbital plane of a satellite. Cosmic Research **3** (1965), pp. 540–553.


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