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# Solving Gyroscopic Eigenvalue Problems with a Real Symmetric Matrix of the Same Dimension

**Abstract** The paper presents an approach for solving the eigenvalue problem of linearised gyroscopic systems. Starting from a system of $n$ second-order differential equations of motion, one is lead to an eigenvalue problem represented by an $n \times n$ Hermitian matrix whose eigenvectors are complex, the elements of the matrix being a function of the frequency parameter. It is first shown that if the Hermitian matrix satisfies certain properties (which define a gyroscopic-system subclass), the problem can be readily transformed into the real eigenvalue problem associated with an $n \times n$ real and symmetric matrix. The physical significance of this transformation is outlined and practical spacecraft applications are quoted to illustrate the approach. The method is then extended to the case of a general Hermitian matrix associated with gyroscopic systems, after reduction of the matrix to an equivalent $2 \times 2$ matrix.

**Résumé** On présente une approche pour résoudre le problème aux valeurs propres dans les systèmes gyroscopiques linéarisés. A partir d'un système de $n$ équations différentielles du second ordre caractérisant le mouvement, on aboutit à un problème aux valeurs propres comportant une matrice hermitienne $n \times n$ dont les vecteurs propres sont complexes, les éléments de la matrice étant fonction du paramètre de fréquence. On montre d'abord que si cette matrice possède certaines propriétés, définissant une sous-classe de systèmes gyroscopiques, on peut facilement se ramener à un problème aux valeurs propres réelles assorties d'une matrice $n \times n$ à la fois symétrique et réelle. On souligne la signification physique de cette transformation et on illustre la méthode par des applications pratiques sur des véhicules spatiaux. On généralise ensuite la méthode au cas d'une matrice hermitienne générale associée aux systèmes gyroscopiques, après avoir réduit cette matrice à une matrice équivalente $2 \times 2$.

## Introduction

The theory for the eigenvalue problem associated with spinning flexible systems has received considerable attention since the advent of spinning spacecraft. The difficulty of the problem stems from the presence of a skew-symmetric matrix in addition to the symmetric matrices that characterise nonspinning systems, so that, in general, the resulting eigenvectors are complex. Much effort has been devoted to date to the reduction of this eigenvalue problem to the standard form that naturally applies to the nonspinning cases, at the expense of doubling the dimension of the system to obtain the real matrices that result from a first-order formulation. This goal seems to have taken precedence over analysis of the underlying physics, so that alternative approaches have attracted little attention.

Unification of the structure of the eigenvalue problem of gyroscopic systems can be achieved by using the second-order form of the differential equations of motion as a starting point for further development. In this way the eigenvalue problem is typified by an Hermitian matrix whose elements are a function of the frequency parameter. It is first shown that this matrix can be transformed directly into a real symmetric matrix of the same dimension if proper partitioning into blocks of real and purely imaginary numbers can be achieved. The physical counterpart of this structure is that the components of the coordinate vector that characterise the system displacements are either in phase or in quadrature for the natural motions of the system. Typical spacecraft applications and other examples are treated to illustrate the theoretical approach.

The above-mentioned transformation is extended to the more general case in which the elements of the Hermitian matrix are full complex numbers. The transformation is established after reduction of the initial $n \times n$ Hermitian matrix to an equivalent $2 \times 2$ Hermitian matrix, which permits the transformation into a $2 \times 2$ real and symmetric matrix in one step.

The mathematical and physical considerations presented here were worked out in the course of the analysis of a practical problem associated with the development of the spacecraft to be flown in 1983 as ESA's contribution to the joint ESA/NASA International Solar-Polar Mission (ISPM). The computational aspects of the method will not be treated here; this particular paper is primarily concerned with presenting the basic ideas underlying a study that is still in progress.

## Statement of the problem

When considering the small, force-free motion in the neighbourhood of equilibrium of a spinning rigid body carrying flexible appendages, the coordinate vector of the system includes six generalised coordinates to describe the motions of the rigid body with respect to inertial space, and a set of deformation variables to depict the oscillations of the appendages. If the latter are considered as a continuum, the equations of motion form a set of ordinary and partial differential equations, the deformation variables being (continuous) functions of space and time.

When spatial discretisation of the flexible appendages is performed or when the corresponding displacement field is represented by assumed functions of space, the partial differential equations transform directly to ordinary differential equations and the Lagrangian formalism provides the linearised equations of motion in the general form (excluding damping):

$$M\,\ddot{q}(t) + G\,\dot{q}(t) + K\,q(t) = 0 \tag{1}$$

where $q=(q_1, q_2, \ldots q_n)^T$ is a set of generalised coordinates, and $M=M^T$, $G=-G^T$, $K=K^T$ are $n \times n$ constant matrices. If the equilibrium is stable, one can seek a periodic solution to Equation (1) in the form

$$q(t) = X\,e^{j\omega t} \tag{2}$$

where $X$ is an $n$-dimensional constant vector with complex elements, and $\omega$ is a real number. By introducing Equation (2) into Equation (1), one obtains

$$H(j\omega)X = 0 \tag{3}$$

with

$$H(j\omega) = -\omega^2 M + j\omega G + K \tag{4}$$

Matrix $H$ is called the 'impedance matrix' of the system.
The condition for the components of $X$ to be nonzero is

$$\det H(j\omega) = 0 \tag{5}$$

Equations (3) and (5) typify the eigenvalue problem associated with Equation (1). In principle, to each eigenvalue $\omega$ calculated from Equation (5) there corresponds one complex eigenvector $X$ given by Equation (3). Due to the symmetry properties of matrices $M$, $G$ and $K$, matrix $H$ is Hermitian; that is

$$H(j\omega) = H^T(-j\omega) \tag{6}$$

This is reflected in the fact that det $H$ is real, so that Equation (5) contains no imaginary part. Equation (6) shows that the eigenvalues occur in pairs of opposite ·al numbers, and from Equation (3) it follows that the corresponding eigenvectors are complex conjugate [1,2].

Hence, supposing that we can calculate the solutions to Equations (3) and (5), the real solution to Equation (1) (corresponding to the physical problem) has the form

$$q(t) = \sum_{\alpha=1}^{n} c_\alpha \left[ X_\alpha^R \cos(\omega_\alpha t + \phi_\alpha) - X_\alpha^I \sin(\omega_\alpha t + \phi_\alpha) \right] \tag{7}$$

where the coefficients $c_\alpha$ and $\phi_\alpha$ are determined by the initial conditions on the coordinate vector, $X_\alpha^R$ and $X_\alpha^I$ being the real and imaginary parts of the complex eigenvector $X_\alpha$.

It should be noted that the mathematical properties of the above eigenvalue problem stem from the fact that complex numbers are introduced as a mathematical tool in Equation (2) to solve the real problem characterised by Equation (1). This procedure allows the independent time variable to be easily eliminated as indicated in Equation (3). In so doing, the physical aspect of the problem is obscured in the formulation of the modal analysis of the system, but the ll significance of the complex eigenvectors obtained in Equation (3) is restored in Equation (7), where their real and imaginary parts are isolated as the coefficients of time functions that are $\pi/2$ out of phase.

Returning to consideration of the system of Equations (3) and (5), we will briefly summarise the classical method of solution. The usual procedure in dynamics is to double the dimension of the coordinate vector, considering the real and imaginary parts of Equation (3) separately in order to deal only with real and constant matrices (independent of $\omega$). This transformation can be effected in different ways. In Reference 3, for instance, the $2n$-dimensional vector

$$z = \begin{bmatrix} \dot{q} \\ q \end{bmatrix}$$

is introduced and the equivalent double-sized system of equations is obtained

$$[A - E_{2n} j\omega] Z = 0$$

where $Z$ is the complex eigenvector corresponding to the coordinate vector $z$, and where

$$A = \begin{bmatrix} -M^{-1}G & -M^{-1}K \\ E_m & 0 \end{bmatrix}$$

$E_m$ being an $m \times m$ unit matrix. Matrix $A$ is a $2n \times 2n$ real matrix and standard computer programs can be used to solve this classical eigenvalue problem. In Reference 4 a different combination of the above matrices is treated. and correspondance is established between the initial $n$-dimensional problem and two real and symmetric $2n \times 2n$ matrices.

In all cases this procedure is systematic and only requires the use of existing computational methods, but the efficiency of the technique is not optimal. From Equation (7) and the definition of vector $z$. if $Z_\alpha^R$ and $Z_\alpha^I$ are the real and imaginary parts of the eigenvector $Z_\alpha$, we have

$$Z_\alpha^R = \begin{bmatrix} -\omega_\alpha X_\alpha^I \\ X_\alpha^R \end{bmatrix} \qquad Z_\alpha^I = \begin{bmatrix} \omega_\alpha X_\alpha^R \\ X_\alpha^I \end{bmatrix}$$

This relationship between the components of the eigenvectors $Z_\alpha$ and $X_\alpha$ is therefore recalculated indirectly $n$ times for an $n$-dimensional eigenvalue problem.

## Simple transformation for special Hermitian matrices

In this section we will restrict ourselves to the class of gyroscopic systems for which the transfer matrix contains, after rearrangement, two diagonal blocks of real numbers and purely imaginary numbers elsewhere, as shown below. This class of system is wider than it may first appear, as illustrated by the practical examples given later.

When the partitioning mentioned above is possible, the impedance matrix can be written in the form

$$H(j\omega) = \begin{bmatrix} A(\omega) & jB(\omega) \\ -jB^T(\omega) & D(\omega) \end{bmatrix} \tag{8}$$

If matrix $H$ is of dimension $n \times n$, the symmetric matrices $A$ and $D$ are of dimension $p \times p$ and $q \times q$, respectively $(q = n - p)$, the dimension of $B$ being $p \times q$.

If one considers the $H$ matrix as a linear operator on the coordinate vector $X$, one can write

$$Y = H X \tag{9}$$

Defining the matrix $T$ with the previous notation as

$$T = \begin{bmatrix} E_p & 0 \\ 0 & -jE_q \end{bmatrix} \tag{10}$$

one can proceed to the following linear coordinate transformation of the system of Equations (9) [1]:

$$X = TX^* \qquad Y = TY^* \tag{11}$$

so as to have

$$Y^* = H^* X^* = T^{-1} H T X^* \tag{12}$$

with

$$H^*(\omega) = \begin{bmatrix} A(\omega) & B(\omega) \\ B^T(\omega) & D(\omega) \end{bmatrix}$$

where the $n \times n$ matrix $H^*$ is real and symmetric. Since the transformation in Equation (12) is a similitude, the eigenvalues of matrices $H$ and $H^*$ are identical.

The eigenvectors $X^*$ of matrix $H^*$ are real and the corresponding eigenvectors $X$ of matrix $H$ are given in Equation (11).

It is therefore immediately apparent from the structure of matrix $T$ that the components of the eigenvector $X$ are either real or purely imaginary numbers. By simple inspection of Equation (7), the meaning of this structure can be perceived: the physical displacements corresponding to the components of the original coordinate vector are either in phase or in quadrature for the natural motions of the system. In this particular case, the total number of nonzero elements in the two vectors $X_x^R$ and $X_x^I$ in Equation (7) cannot exceed $n$ for each mode, which makes the transformation to a $2n$-dimensional system suggested by the classical approach su~erfluous.

## Reduced gyroscopic problem

The technique of reduced impedance matrices will now be briefly described since several of the following examples are based on this approach, which is very convenient when distributed coordinates are used to describe deformations of the system.

If one is primarily interested in the reactions of the central body to appendage deformations in order to study, for instance, the system's attitude stability or pointing accuracy, only that part of the coordinate vector describing the motion of the reference frame must be considered. The number of generalised coordinates then reduces to six if the reference frame is fixed in the central body, and to only three if the reference frame is fixed (inertially) at the centre of mass of the complete structure.

By partitioning the coordinate vector and the corresponding impedance matrix in Equation (3), the deformation coordinates can be eliminated from the equations that describe the motion of the reference frame.

Then

$$H(j\omega)\,X = \begin{bmatrix} A(j\omega) & B(j\omega) \\ C(j\omega) & D(j\omega) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \tag{13}$$

where, in general, the dimensions of vectors $X_1$ and $X_2$ are $p$ and $q=n-p$, respectively. By algebraic elimination of $X_2$, one obtains the following equivalent equation for $X_1$:

$$[A(j\omega) - B(j\omega)\,D^{-1}(j\omega)\,C(j\omega)]\,X_1 = 0 \tag{14}$$

the above matrix being called the 'reduced impedance matrix'.

The order of the system is now reduced to the dimension of $X_1$, independently of the initial dimension. When the eigenvalue problem is solved for the components of $X_1$, the remaining part of the coordinate vector is given by

$$X_2 = -D^{-1}(j\omega)\,C(j\omega)\,X_1 \tag{15}$$

It must be noted that the zeros of the determinant of the reduced impedance matrix give the natural frequencies corresponding to the eigenmodes in which the selected variables participate. In this case, the so-called 'local modes' (deformation of the appendage without reaction of the central body) are eliminated. It is assumed here that all the variables of the initial coordinate vector are coupled through Equation (13), so that no singularity arises in the course of the reduction, the decoupled equations being analysed separately.

When the flexible parts of the system are treated as a continuum, the deformations are described by distributed coordinates which are the solutions of partial differential equations. Equations (13)–(15) remain valid, provided the submatrix symbols are replaced by linear operators[5–7]. The elements of the resulting impedance matrix are generally transcendental functions of $\omega$, whereas in the discrete approach they are polynomial fractions in $\omega$. In all cases the reduced impedance matrix remains Hermitian, which follows from the partitioning in Equations (13) and (14). A specific example of this type is worked out in detail in Reference 8.

## Examples

**An asymmetric, spinning rigid body**

The principal axis frame is chosen as the reference frame. The corresponding inertias are $I_1$, $I_2$ and $I_3$. If the body spins nominally about the third axis with nominal spin velocity $\omega_0$, the Lagrangian formalism provides two coupled linearised equations in the form of Equation (1) for the small angular displacements $\theta_1$ and $\theta_2$ (Tait-Bryan angles) that characterise the nutational motion of the body, with

$$M = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \quad G = \begin{bmatrix} 0 & -\omega_0 (I_1 + I_2 - I_3) \\ \omega_0 (I_1 + I_2 - I_3) & 0 \end{bmatrix}$$

$$K = \begin{bmatrix} \omega_0^2 (I_3 - I_2) & 0 \\ 0 & \omega_0^2 (I_3 - I_1) \end{bmatrix}$$

To each nonzero coefficient in $G$ there corresponds one zero coefficient in $M$ and $K$, so that the corresponding impedance matrix $H$ has the property developed above, in the second section of this paper. Then, introducing the above matrices into Equation (4),

$$H(j\omega) = \begin{bmatrix} (I_3 - I_2)\,\omega_0^2 - I_1 \omega^2 & -j\omega\omega_0 (I_1 + I_2 - I_3) \\ j\omega\omega_0 (I_1 + I_2 - I_3) & (I_3 - I_1)\,\omega_0^2 - I_2 \omega^2 \end{bmatrix}$$

The 2 × 2 transformation matrix

$$T = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}$$

provides, according to Equation (12),

$$H^*(\omega) = \begin{bmatrix} (I_3 - I_2)\,\omega_0^2 - I_1 \omega^2 & -\omega\omega_0 (I_1 + I_2 - I_3) \\ -\omega\omega_0 (I_1 + I_2 - I_3) & (I_3 - I_1)\,\omega_0^2 - I_2 \omega^2 \end{bmatrix}$$

The solution of det $H^*(\omega) = 0$ gives the two eigenfrequencies

$$\omega_1 = \omega_0 \qquad \omega_2 = \omega_0 \sqrt{\left(\frac{I_3}{I_1} - 1\right)\left(\frac{I_3}{I_2} - 1\right)}$$

and if $\theta_1$ and $\theta_2$ represent the components of one eigenvector associated with the above matrix $H$, Equation (11) gives

$$\theta_1^* = \theta_1 \qquad \theta_2^* = j\,\theta_2$$

Since $\theta_1^*$ and $\theta_2^*$ are real numbers, $\theta_1$ is a real number and $\theta_2$ is a purely imaginary number. According to Equation (7), the resulting angular displacements $\theta_1(t)$ and $\theta_2(t)$ must be in quadrature.

The purpose of this trivial example is to illustrate the modal-analysis procedure when advantage is taken of this particular form of the impedance matrix. The following examples are taken from the literature to show the wide variety of practical problems that are compatible with the above simplifying conditions.

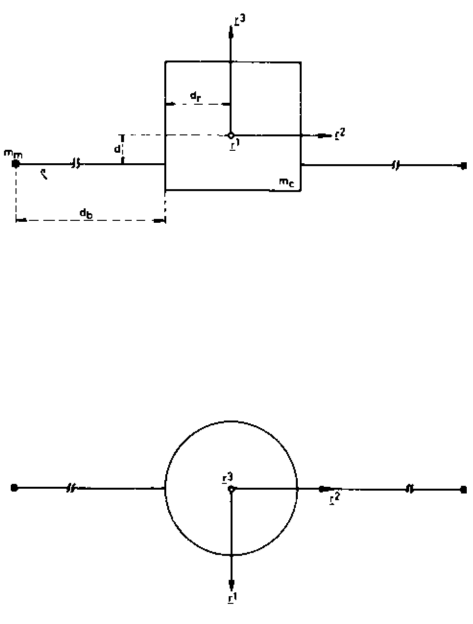


Figure 1

**The Geos satellite (Fig. 1)**

In Reference 6, the reduced impedance matrix of the Geos satellite is given explicitly. This spinning system consists of one asymmetric rigid body to which two 20 m cable booms are attached symmetrically. In the nominal configuration, the cables are parallel to the axis of intermediate inertia of the main body. There is a vertical offset between the centre of mass of the main body and the attachment points of the cables. The matrix is derived on the basis of a continuous approach. The reference frame is the central axis frame of the complete undeformed satellite. In the reduced matrix, only four variables are coupled – $\zeta_1$, $\zeta_2$, $\theta_1$ and $\theta_2$, which describe the lateral oscillations of the central body with respect to the total system's mass centre and its nutational motion. The general form of this matrix with the nominal spin set equal to 1 is as follows:

| | | | |
|---|---|---|---|
| $-(\omega^2+1)(m+f_1)$ | $-2j\omega(m+f_1)$ | $-2j\omega\, d f_1$ | $(\omega^2+1)\, d f_1$ |
| $2j\omega(m+f_1)$ | $-(\omega^2+1)-4f_2$ | $-4\, d f_2$ | $-2j\omega\, d f_1$ |
| $2j\omega\, d f_1$ | $-4\, d f_2$ | $I_3-I_2-I_1\omega^2$ $-(\omega^2-1)g_1-4d^2f_2$ | $-j\omega(I_1+I_2-I_3)$ $-2j\omega\, d^2 f_1$ |
| $(\omega^2+1)\, d f_1$ | $2j\omega\, d f_1$ | $j\omega(I_1+I_2-I_3)$ $+2j\omega\, d^2 f_1$ | $I_3-I_1-\omega^2 I_2$ $-d^2(\omega^2+1)f_1$ |

where $m$ is the mass of the total system. $I_1$, $I_2$ and $I_3$ are the principal inertias of the main body, $d$ is the vertical offset between the cable-attachment points and the main body's centre of mass, and $f_1(\omega)$, $f_2(\omega)$ and $g_1(\omega)$ are transcendental functions of $\omega$ which arise from the solution of the partial differential equations describing the cable-boom motions.

This matrix contains only real and purely imaginary coefficients. The corresponding coordinate vector is $q=(\zeta_1, \zeta_2, \theta_1, \theta_2)^T$. By rearranging the lines (and the columns accordingly) such that $q'=(\zeta_1, \theta_2, \zeta_2, \theta_1)^T$ is the new vector, the particular matricial structure of Equation (8) is obtained and the transformation of the second section applies. The impedance matrix associated with the transformed coordinate vector $q^*=(\zeta_1, \theta_2, j\zeta_2, j\theta_1)^T$ is real and symmetric, as shown below:

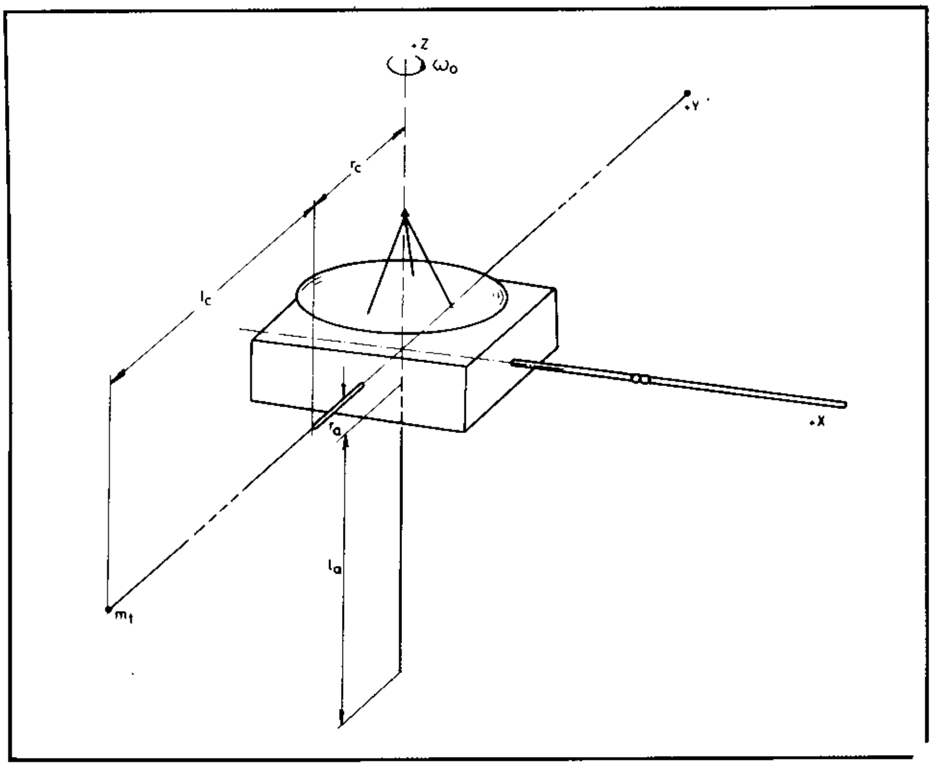
| | | | |
|---|---|---|---|
| $-(\omega^2+1)(m+f_1)$ | $(\omega^2+1)\, d f_1$ | $-2\omega(m+f_1)$ | $-2\omega\, d f_1$ |
| $(\omega^2+1)\, d f_1$ | $I_3-I_1-I_2\omega^2$ $-d^2(\omega^2+1)f_1$ | $2\omega\, d f_1$ | $\omega(I_1+I_2-I_3)$ $+2\omega\, d^2 f_1$ |
| $-2\omega(m+f_1)$ | $2\omega\, d f_1$ | $-(\omega^2+1)-4f_2$ | $-4\, d f_2$ |
| $-2\omega\, d f_1$ | $\omega(I_1+I_2-I_3)$ $+2\omega\, d^2 f_1$ | $-4\, d f_2$ | $I_3-I_2-I_1\omega^2$ $-(\omega^2-1)g_1-4d^2 f_2$ |

**The ISPM spacecraft (Fig. 2)**

In Reference 8 the complete derivation of the reduced impedance matrix for the ISPM spacecraft through the continous approach is given in detail. The spacecraft modelling resembles that for the Geos satellite described in the preceding example, with two major modifications. First, a long axial antenna is cantilevered from the main body along the spin axis; secondly, there is no vertical offset between the attachment points of the cable booms and the centre of mass of the system composed of the central body and the axial antenna. The reference frame is the central axis frame of the complete undeformed satellite.

The re-partition of real and purely imaginary numbers in the impedance matrix corresponding to the coordinate vector defined in the previous section is identical to that of the Geos matrix, although the expressions for the coefficients are more complicated. Setting $d=0$ in the latter and adding the impedance matrix of an axial boom [5] provides the complete matrix for the satellite. The influence of that boom on the central body is reflected in transcendental functions of $\omega$. The considerations for Geos also apply to the treatment of this eigenvalue problem.

Figure 2



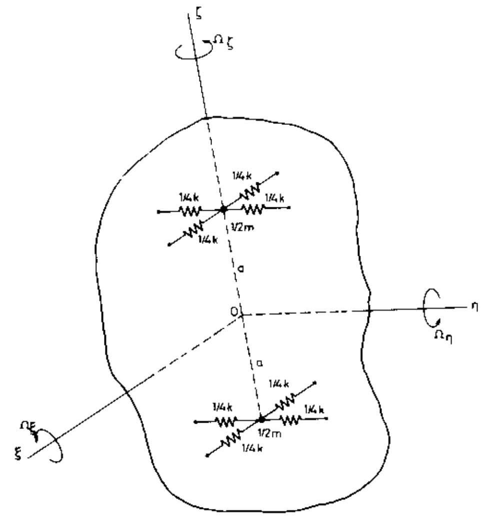




Figure 3

**A typical literature case (Fig. 3)**

In Reference 4 the following example is quoted to illustrate the solution of the eigenvalue problem of gyroscopic systems via a first-order formulation. The system consists of a symmetric spinning rigid body (moments of inertia $A$, $C$, $C$) containing two equal masses $m/2$ lying at distances $\pm a$ from the centre of mass of the rigid body. At equilibrium (spin rate $\omega_0$), the masses are aligned with the spin axis and each is connected to the rigid body by four identical springs (stiffness $k/4$) so that lateral oscillations may occur. To study the antisymmetric motions of the two point masses, the following hybrid coordinate vector is chosen, the reference frame having its third axis along the spin axis:

$$q = (u_1, u_2, w_1, w_2, \omega_1, \omega_2)^T$$

where $u_1$ and $u_2$ are the components in the reference frame of the displacements of the masses in the equatorial plane, $w_1$ and $w_2$ the components of the corresponding inertial velocities, and $\omega_1$ and $\omega_2$ are the components of the rotation vector of the rigid body; the last four are quasi-coordinates. The following first-order formulation is then obtained:

$$I\,\dot{q} + G\,q = 0$$

with

$$I = \begin{pmatrix} k & 0 & 0 & 0 & 0 & 0 \\ 0 & k & 0 & 0 & 0 & 0 \\ 0 & 0 & m & 0 & 0 & ma \\ 0 & 0 & 0 & m & -ma & 0 \\ 0 & 0 & 0 & -ma & A' & 0 \\ 0 & 0 & ma & 0 & 0 & A' \end{pmatrix}$$

$$G = \begin{bmatrix} 0 & -k\omega_0 & -k & 0 & 0 & 0 \\ k\omega_0 & 0 & 0 & -k & 0 & 0 \\ k & 0 & 0 & -m\omega_0 & ma\omega_0 & 0 \\ 0 & k & m\omega_0 & 0 & 0 & ma\omega_0 \\ 0 & 0 & -ma\omega_0 & 0 & 0 & (C-A')\omega_0 \\ 0 & 0 & 0 & -ma\omega_0 & -(C-A')\omega_0 & 0 \end{bmatrix}$$

where $A' = A + ma^2$ is the moment of inertia of the whole system about a transverse axis.

Although a second-order form has been assumed so far for the equations of motion, so that the dimension of the coordinate vector is kept to a minimum, it can be seen in this particular case of first-order formulation that the property explained in the second section can be retrieved by simple transformation of the above vector.

Introducing the new vector

$$q' = -jq$$

The eigenvalue problem of the above system, with previous notations, can be expressed as

$$[-\omega I + jG]\, X' = 0$$

where $X'$ is the eigenvector corresponding to the new vector $q'$. The above impedance matrix is Hermitian and contains only real or purely imaginary numbers. By simply rearranging the columns (and the corresponding lines), the typical structure of Equation (8) is obtained and the transformation of Equation (10) applies. The transformed coordinate vector $q^*$, which can be expressed in terms of the original variables as

$$q^* = (ju_1, jw_2, j\omega_1, -u_2, -w_1, -\omega_2)^T$$

is associated with a real and symmetric matrix. Note that this transformation is not unique as several combinations of lines and columns lead to an equivalent result. Explicitly:

$$\begin{bmatrix} -k\omega & 0 & 0 & -k\omega_0 & -k & 0 \\ 0 & -m\omega & ma\omega & k & m\omega_0 & ma\omega_0 \\ 0 & ma\omega & A'\omega & 0 & -ma\omega_0 & (C-A')\omega_0 \\ -k\omega_0 & k & 0 & -k\omega & 0 & 0 \\ -k & m\omega_0 & -ma\omega_0 & 0 & -k\omega & -ma\omega \\ 0 & ma\omega_0 & (C-A')\omega_0 & 0 & -ma\omega & -A'\omega \end{bmatrix}$$

## Extension to the general case

When the partitioning of the impedance matrix cannot be performed as in Equation (8), which is the case if the matrix contains fully complex numbers (real and imaginary parts non-zero), a different procedure can be adopted to retrieve a real and symmetric matrix without doubling the dimension of the original coordinate vector.

By following the reduction procedure in Equations (13) and (14), it is possible to reduce the original $n \times n$ impedance matrix to a $2 \times 2$ Hermitian matrix. Once the eigenvalue problem has been solved for this reduced system, the complete eigenvectors of the system can be calculated by matrix-multiplication operations, as shown in Equation (15). As noted earlier the modes in which the two selected variables do not participate are eliminated from the solutions of the eigenvalue problem by this technique. In this respect, it is advantageous for spacecraft applications to retain the true coordinates that describe the attitude motions of the reference frame (fixed in the central body). In so doing, the general form of the reduced $2 \times 2$ matrix $H$ is

$$H(j\omega) = \begin{bmatrix} a & c+jb \\ c-jb & d \end{bmatrix} \tag{16}$$

where $a$, $b$, $c$ and $d$ are real functions of the frequency so that the off-diagonal elements are complex. Matrix $H$ is transformed into matrix $H^*$ according to the scheme of Equations (11) and (12), where matrix $T$ must be replaced by the following matrix $S$:

$$S = \begin{bmatrix} \cos\psi & j\sin\psi' \\ \sin\psi & -j\cos\psi \end{bmatrix} \tag{17}$$

where $\psi$ is a parameter. Explicitly,

$$H^*(\omega) = \begin{bmatrix} a\cos^2\psi + d\sin^2\psi + \frac{c}{2}\sin 2\psi & j\left(\frac{a-d}{2}\sin 2\psi - c\cos 2\psi\right) + b \\ b \\ -j\left(\frac{a-d}{2}\sin 2\psi - c\cos 2\psi\right) & a\sin^2\psi + d\cos^2\psi - \frac{c}{2}\sin 2\psi \end{bmatrix} \tag{18}$$

One is now free to choose the parameter $\psi$ such as to cancel the imaginary part of the off-diagonal elements, i.e.

$$\psi = \frac{1}{2}\arctan\frac{2c}{a-d} \tag{19}$$

If this condition is satisfied, the above matrix $H^*$ is real and symmetric, so that the elements $\theta_1^*$ and $\theta_2^*$ of the corresponding eigenvectors are real. The correspondance between the latter and the original variables is given by the transformation in Equation (17),

$$\begin{aligned} \theta_1 &= \theta_1^*\cos\psi + j\,\theta_2^*\sin\psi \\ \theta_2 &= \theta_1^*\sin\psi - j\,\theta_2^*\cos\psi \end{aligned} \tag{20}$$

From this equation it is clear that the eigenvectors associated with matrix $H$ in Equation (16) are complex, so that the phase between their respective components may take on arbitrary values and depends on the particular mode considered. This explains why the transformation matrix $S$ in Equation (17) must depend on the eigenfrequency [through the parameter $\psi$ defined in Equation (19)] in order to provide the new variables $\theta_1^*$ and $\theta_2^*$ which are in phase. The special case of the second section of this paper is retrieved if $\psi = 0$, so that the components of every eigenvector are in quadrature and the transformation is independent of the eigenfrequency.

The connection between the parameter $\psi$ and the geometry of the system will now be examined for a specific example (Fig. 4). We will consider again the case of the Geos satellite with the following modifications: the vertical offset $d$ between the

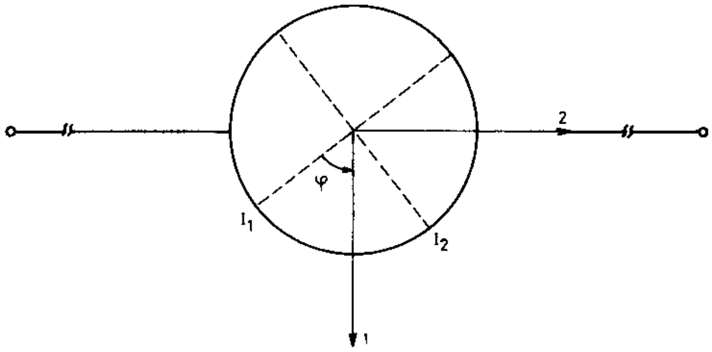



Figure 4

system's centre of mass and the cable booms is zero at equilibrium and the attachment points of the latter are rotated by an angle $\phi$ around the spin axis (axis 3) so that they no longer coincide with the axis of intermediate inertia of the main body. If $d$ is set to zero in the Geos transfer matrix, the last two equations are decoupled from the first two. If the reference frame is also rotated by an angle $\phi$ about the spin axis, so as to be aligned with the cable booms at equilibrium, the following impedance matrix is obtained for the coordinates $\theta_1$ and $\theta_2$ which describe the nutational motion of this reference frame:

| | |
|---|---|
| $(I_3-I_2-I_1\,\omega^2)\cos^2\phi$ $+(I_3-I_1-I_2\,\omega^2)\sin^2\phi$ $-(\omega^2-1)\,g_1$ | $(I_2-I_1)(1-\omega^2)\dfrac{\sin 2\phi}{2}$ $-j\,(I_1+I_2-I_3)\,\omega$ |
| $(I_2-I_1)(1-\omega^2)\dfrac{\sin 2\phi}{2}$ $+j\,(I_1+I_2-I_3)\,\omega$ | $(I_3-I_2-I_1\,\omega^2)\sin^2\phi$ $+(I_3-I_1-I_2\,\omega^2)\cos^2\phi$ |

where $I_1$, $I_2$, $I_3$ are the principal inertias of the main body and $g_1$ is a transcendental function of $\omega$ which reflects the interaction between the cable booms and the central body. The above matrix has the form in Equation (16), so that the corresponding parameter $\psi$, calculated from Equation 19 is

$$\psi = -\frac{1}{2}\arctan\frac{(I_2-I_1)\sin 2\phi}{(I_2-I_1)\cos 2\phi-g_1}$$

On the other hand, the angular position of the principal axis frame of the whole system with respect to the reference frame is given at equilibrium by

$$\psi = -\frac{1}{2}\arctan\frac{(I_2-I_1)\sin 2\phi}{(I_2-I_1)\cos 2\phi-I_c}$$

where $I_c$ the inertia of the two cable booms with respect to the spin axis. Comparing the above formulas, it can be seen that the angle $\psi$ determines the location (with respect to the reference frame) of the principal axis frame of the modified system in which the actual inertia of the cable booms has been replaced by their effective inertia. The latter is associated with the torque exerted by the cables on the main body and depends, of course, on the mode considered. The following special cases are worthy of note: if $\phi=0$ (Geos example) or if $I_1=I_2$ (symmetric body), $\psi=0$ so that no transformation is necessary. If $g_1=0$ (no cable booms), $\psi=-\phi$ which means that for a rigid body the reference axes must be parallel to the principal axes for the impedance matrix not to contain full complex numbers (asymmetric, spinning rigid body example).

## Conclusion

A gyroscopic system of order $n$ has $n$ complex eigenvectors, which in the most general case contain $2n$ unknowns – the relative amplitudes and phases of their components. For a wide class of gyroscopic systems, the components of the eigenvectors are either in phase or in quadrature, which is immediately visible from the structure of the Hermitian impedance matrix associated with the corresponding eigenvalue problem. In this case, this matrix can be transformed into a real symmetric matrix of the same dimension by a straightforward transformation which is independent of the frequency.

In the case where the Hermitian matrix has the most general structure (contains full complex numbers), a simple transformation can be performed on the reduced $2 \times 2$ matrix corresponding to a selected set of convenient variables in order to obtain a $2 \times 2$ real and symmetric matrix. The complete eigenvectors are obtained by matrix-multiplication operations on the eigenvectors of the reduced matrix. The transformation depends implicitly on the frequency parameter.

The advantages of the approach explained in this paper are that the dimension of the system is kept to a minimum and that the physical properties of the modelling are taken into consideration, which in many practical cases simplifies the modal analysis. On the other hand, nonstandard computational problems arise which need further development. For instance, accurate determination of the eigenfrequencies, which are obtained by searching for the zeros of a determinant, may be difficult when the frequencies are close to each other.

The last step needed to achieve generality in the above-mentioned transformation is to elaborate an $n \times n$ unitary matrix which transforms the $n$-dimensional impedance matrix of the most general gyroscopic systems directly (without prior reduction) into an $n \times n$ real and symmetric matrix. This transformation will be treated in a subsequent paper.

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