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Jet Damping and Nutation Growth during the Burn of a Solid Rocket Motor such as PAM-D

Abstract This paper considers the influence of the variation of the mass properties on the transverse dynamics during the burn of a solid rocket motor. Time-dependent Euler equations which express the conservation of the angular momentum for the gas flow and a form of the gas-dynamic torques as given by Flandro are presented. They are integrated for a chosen time history of the mass properties. This model predicts a constant spin rate, a smaller jet damping and an instantaneous nutation frequency above the value based on the instantaneous mass properties. These features are in agreement with flight experience with the PAM-D motor. A nutation growth can be obtained by adjusting the gas-dynamic torques. It remains to be seen whether the predicted nutation growth fits the observed nutation growth.

1. Omega transverse for a time varying system

1.1 General equations of motion

During the burn of a solid rocket motor, the system mass properties vary considerably. To study the impact of this efffect on the transverse dynamics, we consider the following version of the Euler equations at the c.o.m. of the system (Appendix B):

$$A(t)\,\dot{\omega}_1 - [A(t) - C(t)]\omega_2\omega_3 + \dot{m}l(t)^2\,\omega_1 - C(t)[K_1\omega_1 + K_2\omega_2] = 0$$

$$A(t)\,\dot{\omega}_2 + [A(t) - C(t)]\omega_1\omega_3 + \dot{m}l(t)^2\,\omega_2 - C(t)[-K_2\omega_1 + K_1\omega_2] = 0$$

$$C(t)\,\dot{\omega}_3 = 0. \tag{1}$$

The absence of $\dot{A}$, $\dot{C}$ terms as well as the presence of K_1, K_2 will be discussed first. The equations (1) are modification of the dynamical equations obtained on p. 79 of Reference 2. It has been shown that they express the conservation of the total angular momentum for the combustion products between their creation at the burning surface and their exit through the nozzle. As a consequence $\dot{A}_s$ or $\dot{C}_s$ terms are not present in the jet damping term or in the roll equation. From the roll equation follows then immediately:

$$\omega_3(t) = \omega_3(0) = \Omega$$

With $\dot{A}$, $\dot{C}$ terms present, the model predicts that there will always be a spin-up during the burn of a solid rocket motor. For the PAM-D upper stage (Star 48 motor) this spin-up would be substantial. Flight experience[1] shows that the spin rate hardly changes during the burn of these motors. A correct roll equation can only be obtained by using the conservation of the spin component of the angular momentum for the gases (steady flow); this also changes the jet damping terms. It can be argued that the $\dot{A}$ terms should be present in the first two equations of (1) that give the transverse dynamics. This point is still under investigation.

Now the equations for the transverse dynamics can be combined as follows with the complex number $\omega^* = \omega_1 + j\omega_2$:

$$\dot{\omega}^* + \{\, \dot{m}l(t)^2 - C(t)K_1 + j[(A(t) - C(t))\,\Omega + C(t)K_2)]\}\,\frac{\omega^*}{A(t)} = 0 \tag{2}$$

List of symbols and abbreviations

$\dot{m}$	mass flow of the motor (>0 and assumed constant)
$\dot{a}$, $\dot{c}$	rate of change of the transverse, spin moments of inertia of the motor (>0 and assumed constant)
$m_m(t)$	mass of the motor
$a_m(t)$	lateral moment of inertia of the motor
$c_m(t)$	spin inertia of the motor
z_m	distance from reference plane to c.o.m. of motor
m_s	mass of the satellite
a_s	lateral moment of inertia of the satellite (constant)
c_s	spin inertia of the satellite (constant)
$m(t)$	system mass
$A(t)$	lateral moment of inertia of the system (at its c.o.m.)
$C(t)$	spin inertia of the system (at its c.o.m.)
$a'(t)$	transverse inertia of the system at the c.o.m. of the motor
z_n	distance from the separation plane to the exit of the nozzle
z_s	distance from the separation plane to the c.o.m. of the satellite
z_m	distance from the separation plane to the c.o.m. of the motor
z	distance from the separation plane to the c.o.m. of the system
$l(t)$	distance from the c.o.m. of the system to the nozzle exit.
c.o.m.	centre of mass

grouping terms according to the physical effects:

$$\dot{\omega}^* + \{d(t) - K_1\lambda(t) + j[n(t)\Omega + K_2\lambda(t)]\}\ \omega^* = 0$$

$$\text{where } d(t) = \frac{\dot{m}l(t)^2}{A(t)} \text{ jet damping term}$$

$$K_1\lambda(t) = K_1\frac{C(t)}{A(t)} \text{ gas dynamics terms} \tag{3}$$

$$n(t) = 1 - \frac{C(t)}{A(t)} \text{ nutation.}$$

The gas-dynamic terms K_1, K_2 are presented by G. Flandro[7]. They originate from a more detailed flow model for the gases in the motor. Their exact definition is complicated, as they relate to the modal properties of the unsteady vortex flow. A simple physical interpretation is not to hand at the moment. They provide a possible explanation for the nutation growth at the end of the PAM-D burns. In this article they are used as positive constants. This permits the interpretation that they represent an intrinsic property of the motor. In reality, they may depend on system quantities such as $l(t)$, $\dot{A}$. The unit of K_1, K_2 is time^{-1}, their inverse is a time.

The coefficient of ω^* is written as a complex number. The real part will determine the amplitude of $\omega^*(t)$, the imaginary part the frequency. To solve (2) some reasonable assumptions for the evolution of the mass properties during the burn are needed. The model used in this paper is idealised to make the investigation possible. It consists of a system made up of a symmetric satellite with constant mass properties (m_s, a_s, c_s) and a symmetric motor with an invariable c.o.m. and linearly varying mass properties about its c.o.m.. The corresponding time functions for the mass properties are:

$$m(t) = m_m(t) + m_s = m_0 - \dot{m}t \text{ (linear) } (m_0 = m_{m0} + m_s)$$

$$C(t) = c_m(t) + c_s = c_{m0} + c_s - \dot{c}t \text{ (linear)}$$

τ_k	theoretical time to make the mass (inertia) of part k zero based on the mass (inertia) flow of the motor e.g.: τ_m, τ_{mo}, τ_s times corresponding to the total system mass, the motor mass and the satellite mass respectively. Examples for the inertias are τ_c (based on $\dot{c}$),τ_a, (based on $\dot{a}$).
τ_i	$= (a_s + a_m(0))/\ \dot{a}$ (not a physical inertia)
τ_{tr}	$= (z_m - z_s)^2 m_s/\ \dot{a}$ (transfer term)
α, α_s, α_d	intermediate variables
ρ	$\dfrac{z_n - z_s}{z_n - z_m} > 1$
β	$\tau_{mo} + \rho\ \tau_s$
μ	$\dfrac{\tau_{tr}}{\tau_s}\dfrac{1}{(\rho-1)^2}$
ω_i	components of the angular velocity
ω^*	transverse angular velocity as a complex number. $\omega_1+\omega_2=\omega e^{j\phi}$
ω	modulus of ω^*
ϕ	phase angle of ω^*
Ω	spin rate (rad/s)

$$A(t) = a_m(t) + a_s + \frac{m_m(t)}{m(t)} m_s (z_m - z_s)^2 \text{ (quadratic/linear)}$$

$$l(t) = z_n - z(t) \text{ with } z(t) = (z_s m_s + z_m m_m(t))/m(t)$$

$$\text{or } l(t) = \frac{(z_n - z_s)m_s + (z_n - z_m)m_m(t)}{m(t)} \text{ (linear/linear)}$$

Such a time dependency of the mass properties is an acceptable first approximation for the PAM-D upper stage. Other models will be discussed elsewhere[8]. The general solution to the first-order linear differential equation (3) is:

$$\boxed{\omega^* = \psi_0^* \, e^{-X(t)}} \qquad (4)$$

$$\text{where } X(t) = \int_0^t d(u) - K_1\lambda(u) + j\Omega n(u) + jK_2\lambda(u) \, du$$

Denoting

$$X_d(t) = \int_0^t d(u) \, du \; ; \; X_\lambda(t) = \int_0^t \lambda(u) \, du \; ; \; X_n(t) = \int_0^t n(u) \, du$$

$$\text{we have } X(t) = X_d(t) - K_1 X_\lambda(t) + j\omega_{30} X_n(t) + jK_2 X_\lambda(t) \qquad (5)$$

$$\text{and } \boxed{\omega^* = \omega_0^* \, e^{-X_d(t)} \, e^{K_1 X_\lambda} \, e^{-jX_n(t)\Omega} \, e^{-jK_2 X_\lambda}} \qquad (6)$$

Equation (6) contains an amplitude and a phase angle part. The amplitude ω depends on $d(t)$ and $K_1\lambda$ only:

$$\omega(t) = | \, \omega^* \, | \, e^{-X_d(t) + K_1 X_\lambda(t)} \qquad (7)$$

When $d, \lambda > 0$, then X_d, X_λ increase and they have an opposite influence on ω.

The phase function ϕ is $\phi(t) = \Omega \, X_n(t) + K_2 X_\lambda(t) + \phi_0$ (8)

From (8) the instantaneous (nutation) frequency ratio to the spin (denoted $g(t)$) is given as:

$$g(t) = \frac{\phi(t)}{\Omega t} = \frac{X_n}{t} + \frac{K_2 X_\lambda + \phi_0}{\Omega t} \qquad (9)$$

It is this function rather than $n(t)$ that must be considered for the nutation-to-spin ratio in a time-varying system. The function $n(t)$ has this meaning *only when it reduces to a constant,* i.e. for a constant mass system.

The definition of $g(t)$ holds for any $n(t), \lambda(t)$. The value of $g(t)$ is not given by the mass properties *at t only,* but depends on the history (from 0 to t) of the system via the integrals in X_d, X_λ. This is also true when $K_2 = 0$ $g(t)$ is in general different from $n(t)$. In fact, in this case, $g(t)$ is the average value of $n(t)$ up to t and the fluctuations of $n(t)$ are smoothed.

1.2 Phase function ϕ

The first term of the phase function ϕ is X_n and to evaluate it with the aid of the results from Appendix A $\lambda(t)$, we rewrite it as the ratio of two quadratic functions:

$$X_\lambda(t) = \int_0^t \frac{C_s(u)}{A_s(u)} \, du = \int_0^t \frac{\dot{m}\dot{c}N(t)}{\dot{a}\dot{m}D(t)} \, dt$$

where $\dot{c}\dot{m}N(t) = m(t)\ C_s(t) = (\tau_m - t)(\tau_c - t)$

$$\dot{a}\dot{m}\ D(t) = [a_m(t)+a_s]\ m(t) + m_m(t)\ m_s\ (z_m - z_s)^2$$

$$= \dot{a}\dot{m}[t^2 - (\tau_m + \tau_{a'})t + \tau_m\tau_i + \tau_{m0}\tau_{tr}]$$

and $\quad D(t) = (p - t)(q - t)$

The two roots of $D(t)$ are aways real, positive and equal to:

$$p = \tau_{a'}\ \alpha_s - \tau_m\ \alpha_d \cong \tau_{a'}\ (1+\alpha) - \tau_m\ \alpha \tag{10}$$

$$q = \tau_m\ \alpha_s - \tau_{a'}\ \alpha_d \cong \tau_m\ (1+\alpha) - \tau_{a'}\ \alpha \tag{11}$$

where $\alpha = \dfrac{\tau_s\tau_{tr}}{(\tau_{a'} - \tau_m)^2} \quad \alpha_s = (\sqrt{1+4\alpha}+1)/2 \quad \alpha_d = (\sqrt{1+4\alpha} - 1)/2$

The smaller root is $q < \tau_m$.

The results of Appendix A can be used with:

$$c_p = \frac{N(p)}{p-q}\ [T]\ ; \quad c_q = \frac{N(q)}{q-p}\ [T]$$

and $X_\lambda(t)$ can be written as:

$$X_\lambda(t) = \frac{\dot{a}}{\dot{c}} \left[t + c_p\ ln \left(1 - \frac{t}{p}\right) + c_q\ ln \left(1 - \frac{t}{q}\right)\right] \tag{12}$$

The related functions X_n and g_n are:

$$X_n(t) = t - \frac{\dot{c}}{\dot{a}} \left[t + c_p\ ln \left(1 - \frac{t}{p}\right) + c_q\ ln \left(1 - \frac{\tau}{q}\right)\right] \tag{13}$$

$$g_n(t) = 1 - \frac{\dot{c}}{\dot{a}} - \frac{\dot{c}}{\dot{a}t} \left[c_p\ ln \left(1 - \frac{t}{p}\right) + c_q\ ln \left(1 - \frac{t}{q}\right)\right] \tag{14}$$

The equations (10, 11) for p, q give the following expressions for c_p, c_q:

$$c_p = (\tau_{a'} - \tau_m)\ \frac{\alpha_s^{\ 2}}{\alpha_s + \alpha_d} - \frac{\alpha_s}{\alpha_s + \alpha_d}\ (\tau_c - \tau_m) > 0 \tag{15}$$

$$c_q = -(\tau_{a'} - \tau_m)\ \frac{\alpha_d^{\ 2}}{\alpha_s + \alpha_d} - \frac{\alpha_d}{\alpha_s + \alpha_d}\ (\tau_c - \tau_m) < 0 \tag{16}$$

The first term of $g_n(t)$ has the same appearance as $n(t)$ but is based on the change in the inertias instead of on the inertias themselves. The second term in the phase function is due to K_2. The corresponding K_2X_λ follows immediately from the calculations above:

$$K_2X_\lambda = K_2\ \frac{\dot{c}}{\dot{a}} \left[t + c_p\ ln \left(1 - \frac{t}{p}\right) + c_q\ ln \left(1 - \frac{t}{q}\right)\right]$$

The global frequency-ratio function including both the effects of nutation and K_2 is:

$$g(t) = 1 - \left(1 - \frac{K_2}{\Omega}\right) \frac{\dot{c}}{\dot{a}} \left[1 + \frac{c_p}{t}\ ln \left(1 - \frac{t}{p}\right) + \frac{c_q}{t}\ ln \left(1 - \frac{t}{q}\right)\right] \tag{17}$$

When K_2 is small compared with Ω, *only the effects of the changing inertias on g(t) are observable* and make the observed nutation frequency different from nt). The fact that the observed frequency $g(t)$ is different from a calculated $n(t)$ is in itself no evidence for the presence of K_2. Equation (17) must be used to identify K_2.

1.3 Amplitude function ω

The remaining terms $\mathrm{d}(t)$ and $K_1\lambda(t)$ will influence the amplitude of ω^*. Starting with the jet damping, we reorganise $\mathrm{d}(t)$ as follows to facilitate the calculation of X_d:

$$\mathrm{d}(t) = \frac{\dot{m}l^2(t)}{A_s(t)} = \mu \frac{(\beta - t)^2}{(q-t)(\tau_m - t)(p-t)}$$

$$\mu = \frac{\dot{m}}{\dot{a}}(z_n - z_m)^2 = \frac{\tau_{tr}}{\tau_s(\rho-1)^2}$$

$$\beta = \tau_{m0} + \rho\,\tau_s = \tau_m + (\rho-1)\tau_s$$

From Appendix A we obtain immediately $J(t) = \mathrm{e}^{-X_d(t)}$:

$$J(t) = \left(1 - \frac{t}{q}\right)^{e_q} \left(1 - \frac{t}{\tau_m}\right)^{e_t} \left(1 - \frac{t}{p}\right)^{e_p} \tag{18}$$

where

$$e_q = A_q\mu = \frac{(\beta-q)^2}{(q-\tau_m)(q-p)}\,\mu$$

$$e_t = A_t\mu = \frac{(\beta-\tau_m)^2}{(\tau_m-q)(\tau_m-p)}\,\mu$$

$$e_p = A_p\mu = \frac{(\beta-p)^2}{(p-\tau_m)(p-q)}\,\mu$$

Substituting the results for p, q from (10, 11) and grouping terms we have:

$$e_q = \frac{(\sqrt{\alpha_s} + \sqrt{\mu\alpha_d})^2}{\alpha_s + \alpha_d} > 0 \tag{19}$$

$$e_t = -1 < 0 \tag{20}$$

$$e_q = \frac{(\sqrt{\alpha_d} + \sqrt{\mu\alpha_s})^2}{\alpha_s + \alpha_d} > 0 \tag{21}$$

Equation (18) can be written as:

$$J(t) = \frac{\left(1 - \frac{t}{q}\right)^{e_q} \left(1 - \frac{t}{p}\right)^{e_p}}{\left(1 - \frac{t}{\tau_m}\right)} \tag{22}$$

The evolution of the amplitude is no longer described by an exponential decrease, but is as given by the power functions in (22). The denominator tends to increase the amplitude. Normally the effect of the two terms in the numerator dominates ($q < \tau_m$).

Finally, the gas dynamics term due to K_1 also affects the amplitude. From the previous calculations we have immediately:

$$K_1X_1 = K_1 \frac{\dot{c}}{\dot{c}} t + ln \left(1 - \frac{t}{p}\right)^{-\frac{\dot{c}}{\dot{a}}K_1c_q} + ln \left(1 - \frac{t}{q}\right)^{\frac{\dot{c}}{\dot{a}}K_1c_q}$$

$$\text{and } e^{K_1X_\lambda} = e^{\frac{\dot{c}}{\dot{a}}K_1t} \left(1 - \frac{t}{p}\right)^{\frac{\dot{c}}{\dot{a}}K_1c_p} \left(1 - \frac{t}{q}\right)^{\frac{\dot{c}}{\dot{a}}K_1c_q} \tag{23}$$

K_1 contains an increasing exponential term when $K1$ is a positive constant.

$$\text{The corresponding time constant } \tau_k \text{ is: } \tau_k = \frac{\dot{a}}{\dot{c}} K_1^{-1} \tag{24}$$

The exponential term comes from the fact that the numerator and denominator of $\lambda(t)$ are of the same degree in t. For the jet damping function $d(t)$, the degree of the numerator is smaller than the degree of the denominator and no such term is found in (18).

Combining (18) and (23) we have the total amplitude function:

$$\boxed{\omega = \left(1 - \frac{t}{q}\right)^{\epsilon_q} \left(1 - \frac{t}{p}\right)^{\epsilon_p} \frac{e^{t/\tau_k}}{\left(1 - \frac{t}{\tau_m}\right)}} \tag{25}$$

where the exponents of the power functions are given by:

$$\epsilon_q = e_q + \frac{c_q}{\tau_k} \tag{26}$$

$$\epsilon_p = e_p + \frac{c_p}{\tau_k} \tag{27}$$

where e_p, e_q are given by Equations 19 and 21 and c_p, c_q by Equations 15 and 16. From these expressions it is clear that ϵ_q is always positive and that there exists a K_1^* such that for $k_1 > K_1^*$ ϵ_p is negative.

Hence for $K_1 < K_1^*$ the two power functions in (25) decrease, while the exponential and the inverse increase linearly. In the other case, only the power function in ϵ_p decreases.

For small t (beginning of the burn) the amplitude increases or decreases according to s being positive or negative where s:

$$s = \frac{1}{\tau_k} + \frac{1}{\tau_m} - \frac{\epsilon_q}{q} - \frac{\epsilon_p}{p} \tag{28}$$

2. Numerical application to PAM-D

The results of the previous sections will now be applied to the Star-48 motor. Only some simple examples are given. More detailed investigation and comparisons with numerical integration are given in Reference 8. The following approximate data for the STAR-48 motor were reconstituted from Reference 1. These values should not be taken to be exact; we hope that the order of magnitude is representative (in MKS-units): burn time 86 (s); $\dot{m}$ = 23.55 (kg/s); $\dot{a}$ = 4.23 (kgm^2/s); $\dot{c}$ = 3.94 (kgm^2/s); τ_{m0} = 93.64 (s); τ_{amo} = 106.55 (s); τ_{cmo} = 96.55 (s).

Notice that $\dot{a}$ and $\dot{c}$ are almost equal. For a slender motor, not only would c/a be much smaller, but so too would $\dot{c}/\dot{a}$.

$$z_m = -.78\ (m); \quad z_n = -2.1\ (m); \quad \mu = 9.90$$

For the satellite data the following three cases from Reference 1 are considered. The first two (SBS, RCA) are typical spacecraft using PAM-D that showed a fast nutation growth at the end of the burn. No such growth was present on the third one (SGS 1st stage). It is planned to use the PAM-D stage for the ESA spacecraft Ulysses. Its mass properties are quite different from the data used here. Hence extrapolating the observed nutation growth is a delicate matter. Moreover, at one point, the possibility of using PAM-D as an upper stage was envisaged by the Cluster project. The reconstitution of the mass properties from Reference 1 implied some guesswork:

(I) SBS-type	(II) RCA	(III) SGS 1st stage
m_s = 1251 (kg)	m_s = 1081 (kg)	m_s = 3266 (kg)
a_s = 442 (kgm^2)	a_s = 310 (kgm^2)	a_s = 2453 (kgm^2)
c_s = 457 (kgm^2)	c_s = 328 (kgm^2)	c_s = 612 (kgm^2)
z_s = .912 (m)	z_s = .912 (m)	z_s = 1.44 (m)

The corresponding time constants (in seconds) are:

τ_s	=	52.35	45.24	136.68
τ_{cs}	=	115.94	83.21	155.26
τ_{as}	=	104.43	73.24	579.55
τ_{tr}	=	846.16	731.28	3802.94

For the total system we have:

τ_m	=	144.63	135.52	228.96
τ_c	=	212.59	179.86	251.91
$\tau_{a'}$	=	1057.15	910.97	4489.05
ρ	=	2.28	2.28	2.68
β	=	211.74	195.50	485.83

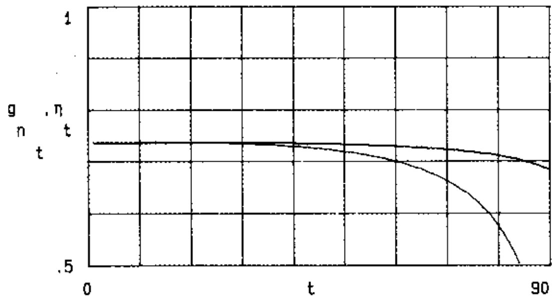
The corresponding derived quantities for the phase function are:

	α	α_s	α_d	p	c_p	p/c_p	q	$-c_q$	$-q/c_q$
(I)	0.053	1.051	0.051	1103.36	849.81	1.298	98.42	5.25	18.75
(II)	0.055	1.053	0.053	951.61	735.05	1.294	96.88	3.94	24.56
(III)	0.029	1.028	0.028	4607.750	4240.870	1.08	110.25	3.74	29.49

The table shows that α is small. Under this condition α_s and α_d can be approximated respectively by $1+\alpha$ and α. The power function in p is very well approximated by the exponental function e^{-t/τ_p} where $\tau_p = p/c_p$. The corresponding frequency-ratio function $g_n(t)$ as given by Equation 14 is plotted in Figures 1—3 and compared with $n(t)$. At the end of the burn, $g_n(t)$ is clearly above $n(t)$ in all cases.

For the amplitude function the following quantities are also needed:

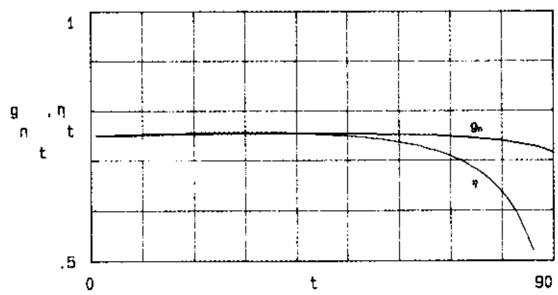
p	e_p	p/e_p	q	e_q	q/e_q
1103.36	8.117	135.935	98.42	2.720	36.184
951.61	8.082	117.74	96.88	2.755	35.169
4607.750	8.598	535.902	110.25	2.239	49.247



$g_{n_{40}} = 0.735566$ $g_{n_{60}} = 0.729586$ $g_{n_{80}} = 0.710922$

$\eta_{40} = 0.72929$ $\eta_{60} = 0.700002$ $\eta_{80} = 0.576709$

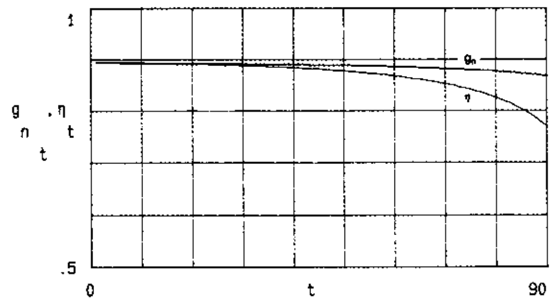
Figure 1. Phase function due to changing inertias (SBS type)



$g_{n_{40}} = 0.754716$ $g_{n_{60}} = 0.752615$ $g_{n_{80}} = 0.739884$

$\eta_{40} = 0.755049$ $\eta_{60} = 0.736876$ $\eta_{80} = 0.636517$

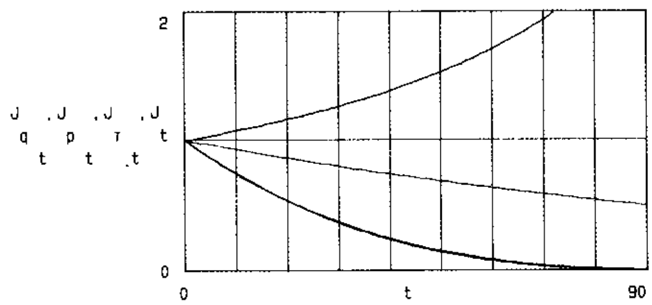
Figure 2. Phase function due to changing inertias (RCA type)



$g_{n_{40}} = 0.890356$ $g_{n_{60}} = 0.885878$ $g_{n_{80}} = 0.877087$

$\eta_{40} = 0.883787$ $\eta_{60} = 0.867863$ $\eta_{80} = 0.82589$

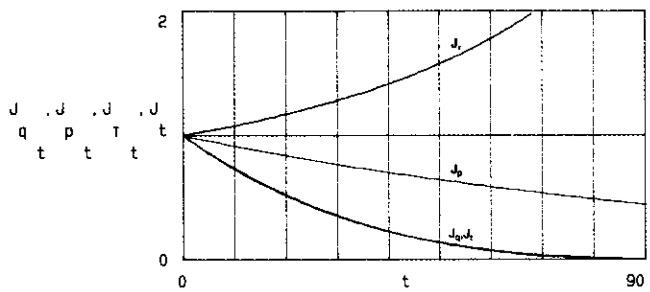
Figure 3. Phase function due to changing inertias (SGS 1st stage)



Jt -> total

$q \cdot \left[1 - e^{\frac{-3}{e^{q}}}\right] = 65.756358$ $J_{67} = 0.050195$

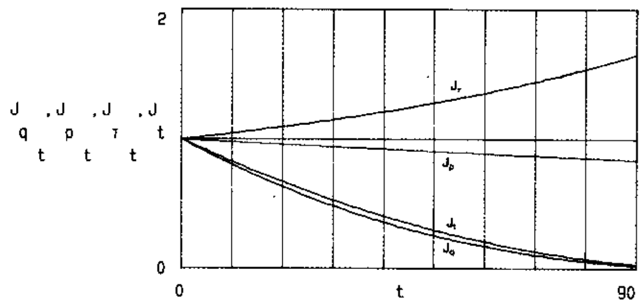
Figure 4. Amplitude function for jet damping only (SBS)



Jt -> total

$q \cdot \left[1 - e^{\frac{-3}{e^{q}}}\right] = 64.277647$ $J_{65} = 0.050107$

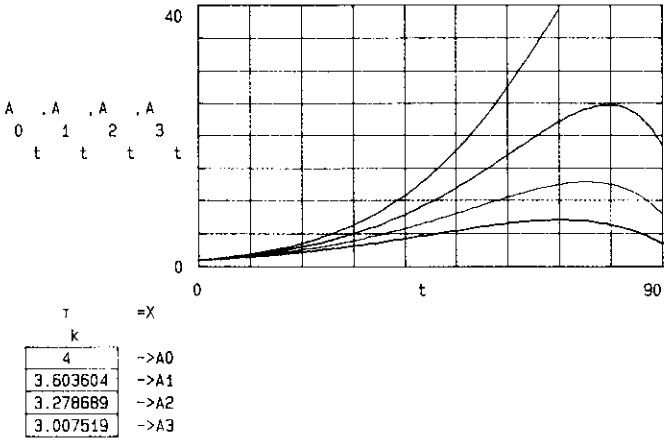
Figure 5. Amplitude function for jet damping only (RCA)



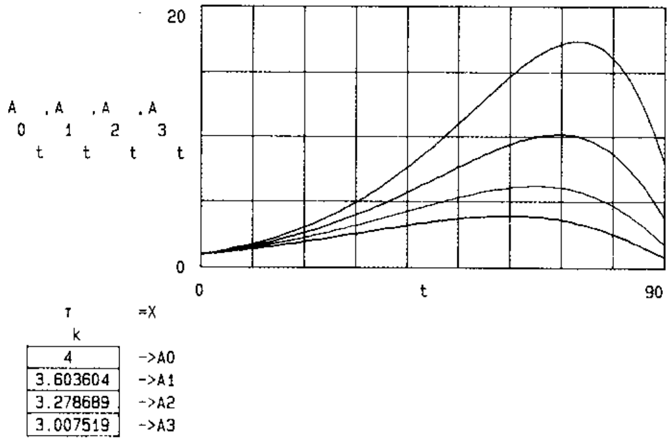
Jt -> total

$q \cdot \left[1 - e^{\frac{-3}{e^{q}}}\right] = 81.383624$ $J_{85} = 0.050001$

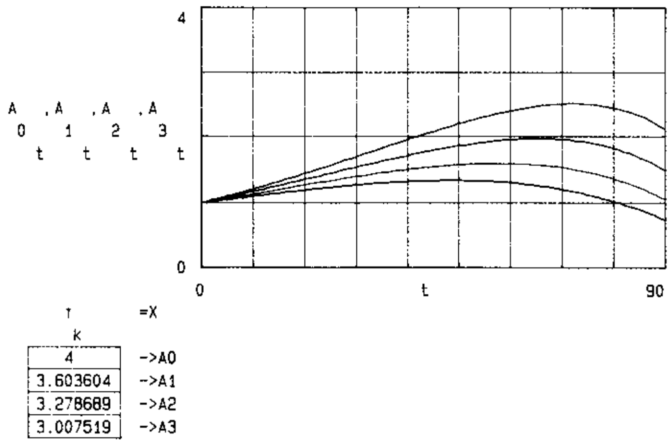
Figure 6. Amplitude function for jet damping only (SGS)



(7)



(8)



(9)

Figure 7. Total amplitude function (SBS)

Figure 8. Total amplitude function (RCA)

Figure 9. Total amplitude function (SGS)

The amplitude function for the jet damping alone is given in Figures 4—6. It takes about 65 s to reduce the initial level to 5% for case I, II. This reduction in efficiency of the jet damping is in agreement with the flight data.

In all three cases the terms due to τ_m and p compensate approximately and the jet damping is well approximated by the q term.

The total amplitude functions are given in Figures 5–8. The value for $\tau_k = \dot{a}/\dot{c}K_1$ must range from 3 to 4 s to have an steep increase of the amplitude at the end of the burn. This implies that the amplitude grows from the beginning of the burn which is not in agreement with the flight data. Moreover, the effect of K_1 is very different in the three cases. The result that case III has a much smaller amplification factor is fine (mostly due to the increase of z_s), but the difference in amplification factors between cases I and II is much too large. This disproves the assumption that K_1 is a constant. Its own time dependency must be found and taken into account. However the structure of Equations (1) remains intact and the results for the phase function and the jet damping are not affected.

3. Conclusions

The influence of the variation of the mass properties on the transverse dynamics during the burn of a solid rocket motor was considered. The time-dependent Euler equations used expressed the conservation of the angular momentum for the gas flow and a form of the gas dynamics torques as given in Reference 7. This model predicts aconstant spin rate, a smaller jet damping and an instantaneous nutation frequency above the value based on the instantaneous mass properties. These features are in agreement with flight experience with the PAM-D motor. The nutation growth computed while assuming K_1 constant does not fit the nutation growth observed during typical PAM-D burns. The time dependency of K_1 must be included. This requires further analysis. Nevertheless one can conclude that the nutation growth is always delayed by a reduction of the ratio of rate of change of the spin to transverse inertia as well as by an increase in the distance between satellite and motor.

Acknowledgment

The author wishes to thank K. Cornelisse, co-author of Reference 2, for his assistance with the starting equations (1) and his colleague E.B. Crellin for fruitful discussions on the subject.

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Appendix A: Integrals

Let $D(t)$ and $N(t)$ be two quadratic functions and p,q be the two real roots of $D(t)$:

then $D(t) = t^2 + b\,t + c = (p - t)\,(q - t)$

and $p = \dfrac{-b + \sqrt{\Delta}}{2}$ $q = -\dfrac{b - \sqrt{\Delta}}{2}$ and $\Delta > 0$

The integral $I_{22} = \int_0^t \dfrac{N(t)}{D}\,dt$ is computed as:

$$I_{22} = t + \int_0^t \frac{N(t) - D(t)}{D(t)}\,dt$$

$$= \int_0^t \frac{-A_p}{p - t} + \frac{-A_q}{q - t}\,dt$$

with $A_p = \dfrac{N(p)}{p - q}$ $A_q = \dfrac{N(q)}{q - p}$

$$\boxed{I_{22} = t + A_p\,ln\left|1 - \frac{t}{p}\right| + A_q\,ln\left|1 - \frac{t}{q}\right|}$$

Next we consider the integral

$$I_{23} = \int_0^t \frac{f_2(t)}{(\tau_1 - t)(\tau_2 - t)(\tau_3 - t)}\,dt$$

where f_2 is a quadratic function of t.

As the degree of the numerator is the degree of the denominator minus one, the function under the integral sign can be expanded as:

$$\sum \frac{A_i}{\tau_i - t}$$

where $A_i = \dfrac{f_2(\tau_i)}{(\tau_i - \tau_j)(\tau_i - \tau_k)} \qquad j,k \neq i$

when the roots τ_i are positive and increasing with i then $A_{1,3} > 0$ and $A_2 < 0$

and $I_{23} = -\ ln\ \Pi \left(1 - \dfrac{t}{\tau_i}\right)^{A_i}$

Appendix B. Equations of motion

We start from the rotation equation as given in (3.4-20) of Reference 2. The total rate of change of the angular momentum dM/dt is equated to the external torques T_{cm}:

$$\int_M \boldsymbol{r}\times\left(\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \boldsymbol{r}\right) \mathrm{d}M + \int_M \boldsymbol{r}\times\{\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})\}\mathrm{d}M +$$
$$2 \int_M \boldsymbol{r}\times\left(\boldsymbol{\Omega} \times \frac{\delta \boldsymbol{r}}{\delta t}\right) \mathrm{d}M + \int_M \boldsymbol{r} \times \frac{\delta^2 \boldsymbol{r}}{\delta t^2} \mathrm{d}M = \boldsymbol{T}_{cm} \qquad \text{(B.1)}$$

where $\boldsymbol{r}$ = the position vector from the instantaneous c.o.m. to an arbitrary point of the system

$\boldsymbol{\Omega}$ = the rotational velocity vector from a frame attached to the c.o.m. of the system;

$\dfrac{\mathrm{d}(\ )}{\mathrm{d}t}$ = the total rate of change of the elemental mass dM

$$= \frac{\delta(\ )}{\delta t} + \Omega \times (\ )$$

$\dfrac{\delta(\ )}{\delta t}$ = the total rate of change with respect to a frame moving with the c.o.m. of the system. When the displacement of the c.o.m. with respect to the rigid part is small, it can be taken as the rate of change with respect to the rigid part of the system. Or $\delta(\ )/\delta t = \boldsymbol{V}_{rp} - \boldsymbol{u}_{cm,rp} \simeq \boldsymbol{V}_{rp}$

In both cases the total (or substantial) derivative is the sum of the local and the convective derivative:

$$\frac{\mathrm{d}(\ )}{\mathrm{d}t} \text{ or } \frac{\delta(\ )}{\delta t} = \frac{\partial(\ )}{\partial t} + /\boldsymbol{v}$$

M = the system under consideration, consisting of a rigid part (motor plus satellite) and the gases in the combustion chamber.

In the absence of external torques (B.1) is rewritten as:

$$\boldsymbol{M}_{in} + \boldsymbol{M}_C + \boldsymbol{M}_{\rm rel} = 0 \tag{B.2}$$

where

$$\boldsymbol{M}_{in} = \int_M \boldsymbol{r} \times \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t} \times \boldsymbol{r}\right) \mathrm{d}M + \int_M \boldsymbol{r} \times \{\Omega \times (\Omega \times \boldsymbol{r})\}\, \mathrm{d}M \tag{B.3}$$

$$M_C = 2 \int_M \boldsymbol{r} \times \left(\Omega \times \frac{\delta \boldsymbol{r}}{\delta t}\right) \mathrm{d}M \tag{B.4}$$

$$M_{\rm rel} = \int_M \boldsymbol{r} \times \frac{\delta^2 \boldsymbol{r}}{\delta t^2}\, \mathrm{d}M = \int_M \frac{\delta}{\delta t}\left(\boldsymbol{r} \times \frac{\delta \boldsymbol{r}}{\delta t}\right) \mathrm{d}M \tag{B.5}$$

M_C, $M_{\rm rel}$ correspond to (4.2—3c,d) with opposite signs as they are part of the total angular momentum and remain on the left-hand side of the equation.

The main problem is to work out the integrals in (B.3—5) under reasonable assumptions for the physical flow behind them. The first term of M_{in} is easily worked out by introducing the inertia tensor:

$$\int_M \boldsymbol{r} \times \left(\frac{\mathrm{d}\Omega}{\mathrm{d}t} \times \boldsymbol{r}\right) \mathrm{d}M \triangleq \boldsymbol{I}\,\frac{\mathrm{d}\Omega}{\mathrm{d}t} \tag{B.6}$$

To work out the second term, we start from the following vector identity:

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + \boldsymbol{B} \times (\boldsymbol{C} \times \boldsymbol{A}) + \boldsymbol{C} \times (\boldsymbol{A} \times \boldsymbol{B}) = 0 \tag{B.7}$$

or

$$\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{C}) + (\boldsymbol{B} \times \boldsymbol{A}) \times \boldsymbol{C} = \boldsymbol{B} \times (\boldsymbol{A} \times \boldsymbol{C}) \tag{B.8}$$

Using this identity with $\boldsymbol{C} = (\boldsymbol{B} \times \boldsymbol{A})$ we have:

$$\boldsymbol{A} \times [\boldsymbol{B} \times (\boldsymbol{B} \times \boldsymbol{A})] + 0 = \boldsymbol{B} \times [\boldsymbol{A} \times (\boldsymbol{B} \times \boldsymbol{A})] \tag{B.9}$$

$$\int_M \boldsymbol{r} \times \{\Omega \times (\Omega \times \boldsymbol{r})\}\, \mathrm{d}M = \int_M \Omega \times \{\boldsymbol{r} \times (\Omega \times \boldsymbol{r})\}\, \mathrm{d}M = \Omega \times (\boldsymbol{I}\Omega) \tag{B.10}$$

$\boldsymbol{M}_{in}$ can be written as:

$$\boxed{\boldsymbol{M}_{in} = \boldsymbol{I}\,\frac{\delta \Omega}{\delta t} + \Omega \times (\boldsymbol{I}\Omega)} \tag{B.11}$$

as it is well known that for the $\boldsymbol{\Omega}$-vector $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t = \delta\boldsymbol{\Omega}/\delta t$

Equation (B.11) is the same as the Euler equation for a rigid, constant mass system although the inertia $\boldsymbol{I}$ is time varying and part of the system has a relative motion.

(B.11) is identical to (4.2—33) of Reference 2 as $\mathrm{d}\boldsymbol{I\Omega}/\mathrm{d}t - \delta\boldsymbol{I\Omega}/\delta t = \boldsymbol{\Omega} \times (\boldsymbol{I\Omega})$ and has been obtained here without using (B.14) (see below).

The Coriolis term M_C will be evaluated via transformations as in [2]. From the chain rule for derivatives we have:

$$\frac{\delta}{\delta t}\{\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r})\} = \frac{\delta \boldsymbol{r}}{\delta t} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) + \boldsymbol{r} \times \left(\frac{\delta \boldsymbol{\Omega}}{\delta t} \times \boldsymbol{r}\right) + \boldsymbol{r} \times \left(\boldsymbol{\Omega} \times \frac{\delta \boldsymbol{r}}{\delta t}\right) \tag{B.12}$$

Adding and subtracting the last term, and using (B.8), we find that (B.12) becomes:

$$\frac{\delta}{\delta t}\{r\times(\Omega\times r)\} = \Omega \times \left(\frac{\delta r}{\delta t}\times r\right) + r \times \left(\frac{\delta\Omega}{\delta t}\times r\right) + 2r \times \left(\frac{\delta r}{\delta t}\right) \quad \text{(B.13)}$$

After integration over M and using the following rule for changing the order of integration and derivation:

$$\int_M \frac{\delta P}{\delta t}\,\mathrm{d}M \triangleq \frac{\delta}{\delta t}\int_M P\,\mathrm{d}M + \int_{A_e} P\,(\rho V.n)\,\mathrm{d}A_e \text{ (for any vector } P) \quad \text{(B.14)}$$

we find that (B.13) becomes:

$$\frac{\delta}{\delta t} I\,\Omega + \int_{A_e} \{r\times(\Omega\times r)\}\,(\rho V.n)\,\mathrm{d}A_e = M_C + I\frac{\delta\Omega}{\delta t} + \Omega \times \int_M \frac{\delta r}{\delta t}\times r\,\mathrm{d}M \quad \text{(B.15)}$$

which gives the following expression for M_C

$$\boxed{M_C = \frac{\delta I}{\delta t}\,\Omega + \Omega \times \int_M \left(4 \times \frac{\delta r}{\delta t}\right)\mathrm{d}M + \int_{A_e} \{r\times(\Omega\times r\}\,(\rho V.n)\,\mathrm{d}A_e} \quad \text{(B.16)}$$

Usually the remaining volume integral is neglected and the surface integral is worked out on the basis of an axial flow model at the nozzle exit.

The second expression of the relative moment M_{rel} (B.5) is rewritten using (B.14):

$$\boxed{M_{\mathrm{rel}} = \frac{\delta}{\delta t}\int_M r\times\frac{\delta r}{\delta t}\,\mathrm{d}M + \int_{A_e} r\times\frac{\delta r}{\delta t}\,(\rho.V.n)\,\mathrm{d}A_e} \quad \text{(B.17)}$$

So far only (B.11), the result for M_{in}, is free of integrals; (B.16-17) for M_C and M_{rel} each still contain one surface and one volume integral.

The surface integrals are worked out as follows:

$$\text{Let } \dot m = -\frac{\mathrm{d}M}{\mathrm{d}t} = \int_{A_e} (\rho V.n)\,\mathrm{d}A_e \quad \text{(B.18)}$$

$$r_{ce} = \frac{1}{\dot m}\int_{A_e} r\,(\rho V.n)\,\mathrm{d}A_e \quad \text{(B.19)}$$

Then (B.18) defines the mass flow $\dot m$ and (B.19) the centre of the mass flow r_{ce} through the nozzle exit. When the motor has rotational symmetry, r_{ce} is on the nominal spin axis.

When decomposing r over A_e as: $r=r_{ce}+r_e$ (B.20)

we have from (B.19): $\int_{A_e} r_e\,(\rho V.n)\,\mathrm{d}A_e = 0$ (B.21)